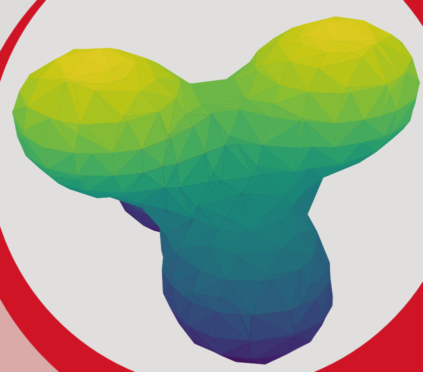
Lecture Notes
Physik

Electrodynamics

Charges, Fields, Relativity and Geometry

BJÖRN MALTE SCHÄFER



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Bibliographic information published by the Deutsche Nationalbibliothek
The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at https://dnb.dnb.de.

Published at Heidelberg University Publishing (heiUP), 2025

Heidelberg University/Heidelberg University Library
Heidelberg University Publishing (heiUP)
Grabengasse 1, 69117 Heidelberg, Germany
https://heiup.uni-heidelberg.de
e-mail: ub@ub.uni-heidelberg.de

The electronic open access version of this work is permanently available on Heidelberg University Publishing's website: https://heiup.uni-heidelberg.de
urn: urn:nbn:de:bsz:16-heiup-book-1055-3
doi: https://doi.org/10.17885/heiup.1055

ISSN 2566-4816
eISSN 2512-4455

ISBN 978-3-96822-170-0 (Softcover)
ISBN 978-3-96822-169-4 (PDF)

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A MAXWELL-EQUATIONS

A.1 *Fields and test charges*

Fields are a very novel concept in physics as there can be systems without direct physical contact of their constituent parts which are nevertheless influenced *at a distance*, as Newton formulated it. As we do not have any direct sensory perception of fields, they are indirectly observed by the force they are capable of exerting on charged test particles. If a test charge q is exposed to the electric field E_i and the magnetic field B^i it experiences a change $\dot{p}_i$ in its momentum p_i:

$$\dot{p}_i = q\left(\mathrm{E}_i + \frac{1}{c}\epsilon_{ijk}\upsilon^j\mathrm{B}^k\right). \tag{A.1}$$

This Lorentz-force depends on the magnitude and direction of the velocity υ^i/c in units of a velocity scale c. It is possible to measure all components of E_i and B^i separately as one has the freedom of choosing the state of motion of the test charge. Clearly, as the velocity υ^i depends on the choice of frame, the measurement of E_i and B^i has to be frame-dependent as well. Therefore, with the concept of a test charge one links the dynamical and kinematical properties of fields to the mechanics of the test particles in a consistent way. Historically this was very important, as electrodynamics showed that Galilean mechanics for the motion of test particles is inconsistent with the fields, and needed to be replaced by Lorentzian, relativistic mechanics. It is important to realise that the two fields measured by a test charge are a linear form E_i for the electric component and a vector B^i for the magnetic component.

A.2 *Physical properties of the electric charge*

Electrodynamics is a continuum theory: One can imagine the electric charge density ρ to be a fluid so that arbitrarily small volumes contain arbitrarily small amounts of electric charge. There is no idea of charge carriers such as electrons or protons, and no concept of a quantisation of charge into multiples of an elementary charge. Charge is conserved, meaning that the fluid can move and change the charge density, but there is no spontaneous creation or annihilation of electric charge. This statement is necessarily an empirical property of charge-carrying matter.

If the local charge density increases, it must be necessarily due to converging current densities, as expressed in a continuity equation:

$$\partial_t\rho + \partial_i\jmath^i = 0 \tag{A.2}$$

implying that the charge q contained within a volume V only changes over time if there are electric currents I transporting the charge through the surface $\partial\mathrm{V}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathrm{V}}\mathrm{dV}\,\rho = \frac{\mathrm{d}}{\mathrm{d}t}q = -\int_{\mathrm{V}}\mathrm{dV}\,\partial_i\jmath^i = -\int_{\partial\mathrm{V}}\mathrm{dS}_i\,\jmath^i = -\mathrm{I} \tag{A.3}$$

as a consequence of the Gauß-theorem. The electric charge q appears as the volume integral over the charge density ρ, in the same way as the electric current I is the surface integral over the current density $\jmath^i$, projected along the surface normal dS_i. Specifically, if the currents $\jmath^i$ point outwards and are parallel to the surface normals

dS_i of ∂V, the enclosed charge q will decrease, which provides a good motivation for the minus-sign.

A.3 *Maxwell-equations*

The Maxwell-equations are the axiomatic foundation of classical electrodynamics and they define the relation between the distribution of electric charges and currents on one side and the electric and magnetic fields on the other, as well as the dynamical evolution of the fields themselves. They are a set of linear hyperbolic partial differential equations, formulated in terms of first derivatives ∂_i and ∂_{ct} of the fields with respect to the coordinates x^i and time t. Temporal derivatives always appear multiplied with a constant c, that will turn out to be the speed at which excitations in the electromagnetic field propagate.

Maxwell's equations involve the two physically measurable fields E_i and B^i as well as two auxiliary fields D^i and H_i. These auxiliary fields are sourced by the electric charge density ρ and the electric current density $\jmath^i$, and can only be related to E_i and B^i with an assumption on the physical properties of the medium in which the charges reside. At the time, Maxwell isolated his four equations from empirical observation of magnetic and electric phenomena, but they are much more than that: They open a path to a geometric description of Nature in terms of relativistic field theories.

A.3.1 Gauß-law for electric fields

The electric field D^i emanates from the electric charge density ρ, meaning that wherever there are electric charges, they act as sources of the electric field. The field lines diverge from a positive charge and converge on a negative charge. Mathematically speaking, the divergence $\partial_i D^i$ of the electric field is proportional to the charge density ρ with the prefactor 4π in the Gauß-system of units:

$$\partial_i D^i = 4\pi\rho \tag{A.4}$$

With the help of the Gauß-theorem the Maxwell-equation can be reformulated in integral form,

$$\int_V dV\, \partial_i D^i = \int_{\partial V} dS_i\, D^i = \psi = 4\pi \int_V dV\, \rho = 4\pi q \tag{A.5}$$

implying that there is a flux ψ of electric field lines through the surface ∂V of any volume V which contains a charge q.

An electric field D^i should be spherically symmetric around a point charge of magnitude q, certainly in the case of an isotropic medium. This means that the electric field lines should be perpendicular to the surface ∂V of a sphere of volume V containing the charge at the centre, and the electric field should be of equal strength everywhere on the surface. Then,

$$\int_{\partial V} dS_i\, D^i = 4\pi\, r^2\, D = 4\pi q \quad\rightarrow\quad D = \frac{q}{r^2} \tag{A.6}$$

with the familiar expression for the Coulomb-field $D \propto 1/r^2$ of a point charge.

In summary, electric field lines start necessarily on a positive charge and end at a negative charge, unless they form a closed loop.

A.3.2 Non-existence of magnetic charges

The magnetic field B^i behaves differently: There are no corresponding magnetic charges from which the magnetic field lines would emanate, so the divergence of the magnetic field is necessarily zero,

$$\partial_i B^i = 0. \tag{A.7}$$

In integral form the relation would read

$$\int_V \mathrm{d}V\, \partial_i B^i = \int_{\partial V} \mathrm{d}S_i\, B^i = \phi = 0 \tag{A.8}$$

showing clearly with the Gauß-theorem that the flux ϕ of magnetic field lines across the surface ∂V of a volume V is zero, as it can not contain any magnetic charges.

A.3.3 Faraday-law and induction

Electric field lines can be closed loops, too, and this is necessarily related to time-varying magnetic fields, as formulated by the Faraday-law

$$\epsilon^{ijk}\partial_j E_k = -\partial_{ct} B^i \tag{A.9}$$

with the most famous minus sign of physics: the Lenz-rule. It is a reflection of the hyperbolicity of the Maxwell-equations and despite many claims otherwise, it has little to do with energy conservation. Here, the speed of light c makes sure that the derivative ∂_i is dimensionally consistent to the derivative ∂_{ct}, because

$$\partial_{ct} = \frac{1}{c}\partial_t \tag{A.10}$$

has units of inverse length just as the spatial derivatives: Please keep in mind, that in the Gauß-system of units, all fields E_i, D^i, H_i and B^i have identical units. The corresponding integral form of the Faraday-law is derived by application of the Stokes-theorem,

$$\int_S \mathrm{d}S_i\, \epsilon^{ijk}\partial_j E_k = U = \int_{\partial S} \mathrm{d}r^i\, E_i = -\frac{\mathrm{d}}{\mathrm{d}(ct)}\int_S \mathrm{d}S_i\, B^i = -\frac{\mathrm{d}\phi}{\mathrm{d}(ct)} \tag{A.11}$$

where one can identify the induced voltage U on the boundary ∂S of the surface S as being proportional to the rate of change of the magnetic flux ϕ with respect to ct. Integrating the rotation $\epsilon^{ijk}\partial_j E_k$ along an integration contour would yield the displacement work necessary to move a charge along this contour, which, normalised by the magnitude of the charge, is exactly the voltage.

A.3.4 Ampère-law

Magnetic fields are surely divergence-free, but can they be loops? Ampère's law answers this clearly in a positive way, as

$$\epsilon^{ijk}\partial_j \mathrm{H}_k = +\partial_{ct}\mathrm{D}^i + \frac{4\pi}{c}\jmath^i \tag{A.12}$$

implying that the rotation $\epsilon^{ijk}\partial_j \mathrm{H}_k$ is related to two phenomena: There can be a non-vanishing electric current density $\jmath^i$ with magnetic field lines looping around it, or the electric field is time-varying. Again, the Stokes-theorem allows to reformulate the Ampère-law in integral form,

$$\int_S \mathrm{dS}_i\, \epsilon^{ijk}\partial_j \mathrm{H}_k = \int_{\partial S} \mathrm{d}r^i\, \mathrm{H}_i = +\frac{\mathrm{d}}{\mathrm{d}(ct)}\int_S \mathrm{dS}_i\, \mathrm{D}^i + \frac{4\pi}{c}\int_S \mathrm{dS}_i\, \jmath^i = +\frac{\mathrm{d}\psi}{\mathrm{d}(ct)} + \frac{4\pi}{c}\mathrm{I} \tag{A.13}$$

such that magnetic field collected up on a closed loop ∂S becomes equal to the change of the electric flux ψ through the surface S and to the electric current I through that surface. In a static, cylindrically symmetric situation of a straight wire one would evaluate the integral as $\int \mathrm{d}r^i \mathrm{H}_i = 2\pi r\,\mathrm{H}$ on a circle with radius r, such that the magnetic field decreases $\mathrm{H} \propto 1/r$ with increasing distance from the wire.

A.4 *Linear media for electrodynamics*

Maxwell's equations allow to compute the electric and magnetic fields for a given distribution of the charge density ρ and the current density $\jmath^i$, and to localise the source distribution ρ and $\jmath^i$ for a given field configuration. In the general case, these relationships are defined for two auxiliary fields, an electric vector field D^i and a magnetic linear form H_i. These two excitations, D^i and H_i, are related to the sources ρ and $\jmath^i$ in purely geometric relations. For converting them into the measurable fields E_i and B^i, one needs to incorporate the properties of matter, in which the charges and currents as sources of the electric and the magnetic field are embedded.

	vector	*form*
vacuum	B^i	E_i
medium	D^i	H_i

Restricting the discussion to linear media one assumes a proportionality

$$\mathrm{D}^i = \epsilon^{ij}\mathrm{E}_j \quad \leftrightarrow \quad \mathrm{E}_i = \epsilon_{ij}\mathrm{D}^j \tag{A.14}$$

with a permissivity (or dielectric) tensor ϵ_{ij}, and in analogy a likewise linear relation

$$\mathrm{B}^i = \mu^{ij}\mathrm{H}_i \quad \leftrightarrow \quad \mathrm{H}_i = \mu_{ij}\mathrm{B}^j \tag{A.15}$$

with a permeability tensor μ_{ij}. These two relationships between the electric field pair E_i and D^i on one side and the magnetic field pair B^i and H_i on the other are referred to as constitutive relations. The permissivity tensor ϵ_{ij} and the permeability tensor μ_{ij} are both symmetric, positive definite tensors. As such, they act as a metric with the purpose of converting the vectors in linear forms and vice versa. This is made possible by the fact that ϵ^{ij} is inverse to ϵ_{ij}, with $\epsilon^{ij}\epsilon_{jk} = \delta^i_k$. Both tensors have a principal axis frame in which one observes that the fields become elementwise proportional to each other, scaled by the eigenvalues of ϵ_{ij} or μ_{ij}.

If all eigenvalues are equal, the medium is isotropic and the tensors become proportional to the Euclidean metric γ_{ij}, with an admittedly weird convention

$$\epsilon_{ij} = \frac{1}{\epsilon}\gamma_{ij} \quad \text{and} \quad \mu_{ij} = \frac{1}{\mu}\gamma_{ij}, \tag{A.16}$$

with the dielectric constant ϵ and the isotropic permeability μ. Vacuum is effectively described to be a medium with $\epsilon = 1 = \mu$. Because the Euclidean metric γ_{ij} mediates between the pairs E_i, D^i and H_i, B^i, their distinctiveness is lost in a vacuum situation without a medium. Positive definiteness makes sure that the observable electric and magnetic fields B^i and E_i are pointing in the same direction as the excitations H_i and D^i: The fields are attenuated in a medium but never reversed.

There are even bianisotropic media where dielectric effects are caused by magnetic fields and effects of permeability by electric fields:

$$D^i = \epsilon^{ij}E_j + \xi^{ij}H_j \quad \text{as well as} \quad B^i = \mu^{ij}H_j + \zeta^{ij}E_j, \tag{A.17}$$

with two additional tensors ξ^{ij} and ζ^{ij} in the constitutive relations. The connection between fields and the microscopic structure of matter can be extremely complicated, and only in simplified cases one will have a linear, instantaneous and isotropic response of the fields to the presence of matter. It is quite apparent that there is a time scale involved in the reaction of the fields E_i and B^i to the excitations H_i and D^i. Water, for instance, has a very high dielectric constant $\epsilon \simeq 80$ for static electric fields as a consequence of the polarity of the water molecules, but the dielectric constant for the rapidly changing electric field in visible light has decreased to a value of about $\epsilon \simeq 1.5$.

In contrast to the inhomogeneous equations that get modified in the presence of matter, the homogeneous relations $\partial_i B^i = 0$ and $\epsilon^{ijk}\partial_j E_k = -\partial_{ct}B^i$ are unchanged in matter. One can almost feel why this should be the case: The inhomogeneous equations predict, due to their shape, strong fields close to the sources, i.e. $D \propto 1/r^2$ and $H \propto 1/r$ which are diverging in the limit $r \to 0$ and which should evoke a strong response from the medium altering the fields. While this argument sounds very convincing, it neglects the fact that most materials follow linear relationships between the fields and the excitations, or equivalently, have constant dielectric and permeability tensors, and effectively do not distinguish between weak and strong fields: In fact, the situation is very puzzling to interpret as there are linear responses evoked by arbitrarily weak fields around sources, but not for induced fields!

For the purpose of this script we will always assume homogeneous media, such that the material properties do not change as a function of position and all derivatives $\partial_k\epsilon^{ij}$ and $\partial_k\mu^{ij}$ are zero. But this is a choice of convenience, as the Maxwell-equations would be prepared to deal with inhomogeneous media, at the expense of notational clarity: The Gauß-law $\partial_i D^i = 4\pi\rho$, for instance, becomes $\partial_i(\epsilon^{ij}E_j) = \partial_i\epsilon^{ij} \cdot E_j + \epsilon^{ij}\partial_i E_j = 4\pi\rho$, with a new term $\partial_i\epsilon^{ij}$ reflecting the change of the dielectric tensor with coordinate.

A.5 *Conservation of electric charge*

With the link between the fields and the distribution of charges (and currents) established by the Maxwell-equations it should be clear that the fields are not free and that they should reflect any dynamical laws to which the charges are subjected, for instance charge conservation: It will never be the case that the electrical field lines suddenly converge onto a single point in space and form a non-vanishing divergence, without there being an actual charge. Therefore, it should be possible to derive a charge conservation law from the field configuration! In fact, computing the divergence of the rotation $\epsilon^{ijk}\partial_j H_k$ as defined by the Ampère-law,

$$\epsilon^{ijk}\partial_j H_k = +\partial_{ct} D^i + \frac{4\pi}{c} \jmath^i \quad \rightarrow \quad \partial_i \epsilon^{ijk}\partial_j H_k = \epsilon^{ijk}\partial_i\partial_j H_k = \partial_i\partial_{ct} D^i + \frac{4\pi}{c}\partial_i \jmath^i = 0 \tag{A.18}$$

which is always zero: The contraction of ϵ^{ijk} which is antisymmetric in the index pair (ij) with the double derivative $\partial_i\partial_j$, which is symmetric in the index pair is necessarily zero. The exchangibility of the partial derivatives $\partial_i\partial_j = \partial_j\partial_i$ and hence the symmetry of the expression $\partial_i\partial_j$ is made sure by Schwarz's theorem.

This consideration leads to

$$\partial_{ct}\partial_i D^i + \frac{4\pi}{c}\partial_i \jmath^i = 0, \tag{A.19}$$

and by substituting the Gauß-law $\partial_i D^i = 4\pi\rho$ to

$$4\pi\partial_{ct}\rho + \frac{4\pi}{c}\partial_i \jmath^i = 0, \tag{A.20}$$

where ∂_{ct} interchanges with the divergence, as both are partial derivatives. In the last relation one recovers the conservation law in the shape of a continuity equation

$$\partial_t\rho + \partial_i \jmath^i = 0 \tag{A.21}$$

In a very real sense, electrodynamics is the theory of electric and magnetic fields for conserved charges; if, by any mechanism, there would be spontaneous creation or decay of charges, or even teleportation of charges, the Maxwell-equations would need to be amended.

A.6 *Electromagnetic duality*

In vacuum, where the charge density ρ and the current density $\jmath^i$ are zero, the Maxwell-equations assume a very symmetric shape as all equations are purely homogeneous. The divergences read

$$\partial_i D^i = \epsilon^{ij}\partial_i E_j = \epsilon\gamma^{ij}\partial_i E_j = 0 \quad \text{as well as} \quad \partial_i B^i = \mu^{ij}\partial_i H_j = \mu\gamma^{ij}\partial_i H_j = 0, \tag{A.22}$$

and the rotations become:

$$\epsilon^{ijk}\partial_j E_k = -\partial_{ct} B^i = -\mu^{ij}\partial_{ct} H_j = -\frac{1}{\mu}\gamma^{ij}\partial_{ct} H_j \tag{A.23}$$

and

$$\epsilon^{ijk}\partial_j H_k = +\partial_{ct} D^i = +\epsilon^{ij}\partial_{ct} E_j = \epsilon\gamma^{ij}\partial_{ct} E_j, \tag{A.24}$$

where we introduced the permissivity and permeability tensors to map all fields to the two linear forms E_i and H_i. In isotropic media, Maxwell's equation exhibit invariance under the duality transform

$$E_i \rightarrow +H_i \quad \text{and} \quad \mu H_i \rightarrow -\epsilon E_i. \tag{A.25}$$

Clearly, there is no influence of the duality transform on the divergences, while the two equations involving rotations just interchange their roles. Duality is broken by the presence of ρ and $\jmath^i$. Without a medium, i.e for the case $\epsilon = 1 = \mu$, the duality transform takes on an even simpler form, $E_i \to \gamma_{ij}B^j$ and $B^i \to -\gamma^{ij}E_j$, and relates the vacuum fields directly to each other.

Maxwell's equations would straightforwardly be able to accommodate magnetic charges if they are in fact conserved, as can be seen from this argument. Purely driven by analogy and intuition, one can amend Maxwell's equations as

$$\partial_i D^i = 4\pi\rho \quad \text{and} \quad \partial_i B^i = 4\pi\tau \tag{A.26}$$

as well as

$$\epsilon^{ijk}\partial_j H_k = +\partial_{ct}D^i + \frac{4\pi}{c}\jmath^i \quad \text{and} \quad \epsilon^{ijk}\partial_j E_k = -\partial_{ct}B^i - \frac{4\pi}{c}\imath^i \tag{A.27}$$

by introducing a magnetic charge density τ and a magnetic current density $\imath^i$, making all Maxwell-equations inhomogeneous PDEs. Clearly, with vanishing τ and $\imath^i$ one would recover the true Maxwell-equations, one pair being homogeneous and the other pair being inhomogeneous. From $\epsilon^{ijk}\partial_i\partial_j E_k = 0$ one recovers a continuity equation of the magnetic charge

$$\partial_t\tau + \partial_i\imath^i = 0 \tag{A.28}$$

in complete analogy to the case for electric charges. This realisation is quite sensible: There are 2 scalar and 2 vectorial equations for 3 components for E_i and 3 components for B^i. They have to be determined by ρ as a scalar source and by $\jmath^i$ as a vectorial source, which might look odd, as there are more field components than source components ($6 > 4$), and again more equations than field components ($8 > 6$)! But the conservation of the source needs to be respected by the fields as well, reducing the effective number of equations by two: there is a conservation law for ρ and one for τ, which Nature has incidentally chosen to be zero (She has good reasons for doing so!), reducing the effective number of equations from 8 to 6. How exactly the 4 components of the source determine 6 components of the fields (clearly, they can't all be independent, otherwise the problem would be underdetermined) will be the topic of Sect. B on potential theory.

A summary of all quantities appearing in the Maxwell-equations is given in this diagram Fig. 1, for the general, hypothetical case of both magnetic and electric charges. For the actual Maxwell-theory with only electric charges, $\tau = 0 = \imath^i$.

A.7 *Maxwell-equations under discrete symmetries*

The Maxwell-equations show a curious and interesting behaviour under the three discrete symmetries: (*i*) charge conjugation $\mathcal{C}$, which replaces every positive charge $+q$ by a negative one $-q$, and vice versa, (*ii*) parity inversion $\mathcal{P}$, which mirrors the spatial coordinates $+x^i$ to $-x^i$, and (*iii*) time reversal $\mathcal{T}$, which replaces $+t$ by $-t$. Particularly relevant will be the classification of vectors (and linear forms) as being polar, $\mathcal{P}D^i = -D^i$ or axial, $\mathcal{P}B^i = +B^i$. Under the assumptions of a linear medium, the two pairs of fields will always be proportional to each other, $E_i = \epsilon_{ij}D^j$ and $H_i = \mu_{ij}B^j$ and must have pairwise identical behaviour under $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$.

Starting from the realisation that the position x^i behaves like a polar vector because its sign change under $\mathcal{P}$ leads to the implication that the differentiation ∂_i

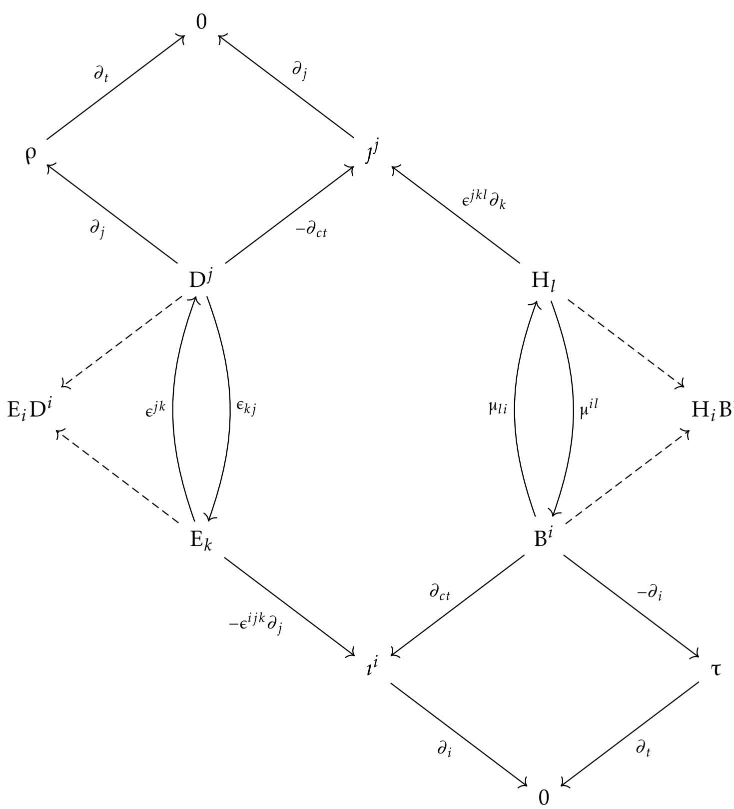


Figure 1: All quantities and their relationships within the Maxwell-equations.

behaves as $\mathcal{P}\partial_i = -\partial_i$. An identical argument applies to time reversal, leading to $\mathcal{T}\partial_{ct} = -\partial_{ct}$ for the time derivatives. The volume needed for computing the densities ρ and τ enters in an unoriented way, so it is unaffected by $\mathcal{P}$. The densities do change sign under $\mathcal{C}$, though. The currents $\jmath^i$ and $\imath^i$ change sign under $\mathcal{C}$ and reverse their direction of flow under both $\mathcal{P}$ and $\mathcal{T}$.

A good starting point are the third and fourth Maxwell-equations,

$$\epsilon^{ijk}\partial_j \mathrm{E}_k = -\partial_{ct}\mathrm{B}^i - \frac{4\pi}{c}\imath^i, \quad \text{and} \quad \epsilon^{ijk}\partial_j \mathrm{H}_k = +\partial_{ct}\mathrm{D}^i + \frac{4\pi}{c}\jmath^i \tag{A.29}$$

with the (possible) extension to include a (conserved) magnetic charge density τ and its associated magnetic current density $\imath^i$. They suggest that D^i and $\jmath^i$ on one side and B^i and $\imath^i$ on the other must have identical properties under the discrete symmetry transformations $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$. But at the same time it is clear that there is a fundamental difference in the behaviour of the electric and magnetic fields with respect to $\mathcal{P}$, as the right hand sides acquire additional minus signs because of the derivative ∂_i: Parity transforms affect electric and magnetic fields in opposite ways. Because the electric fields result from the gradient of a potential, $\mathrm{E}_i = -\partial_i\Phi$, they must be behave as polar vectors, $\mathcal{P}\mathrm{D}^i = -\mathrm{D}^i$, and the magnetic fields as axial vectors, $\mathcal{P}\mathrm{B}^i = +\mathrm{B}^i$.

The two divergences

$$\partial_i \mathrm{D}^i = 4\pi\rho \quad \text{as well as} \quad \partial_i \mathrm{B}^i = 4\pi\tau \tag{A.30}$$

make sure that the fields change sign under $\mathcal{C}$ along with the changes of the charges ρ and τ under $\mathcal{C}$. Far more interesting is $\mathcal{P}$: Because $\partial_i\mathrm{D}^i$ is parity-even, ρ must be scalar, $\mathcal{P}\rho = \rho$, but conversely, $\mathcal{P}\partial_i\mathrm{B}^i = -\partial_i\mathrm{B}^i$ implies a pseudoscalar magnetic charge $\mathcal{P}\tau = -\tau$. This translates to a more subtle difference in the transformation property of the currents $\jmath^i$ and $\imath^i$: The latter needs to be parity positive, $\mathcal{P}\imath^i = +\imath^i$ and therefore axial, while $\mathcal{P}\jmath^i = -\jmath^i$, with a polar electric current density, effectively ensuring the consistency of the two rotational Maxwell-equations.

The two conservation equations $\partial_t\tau + \partial_i\imath^i = 0$ and $\partial_t\rho + \partial_i\jmath^i = 0$ are likewise consistent because $\mathcal{T}$ changes both the time-derivatives as well as the direction of the currents, and parity inversion $\mathcal{P}$ changes τ because of its pseudoscalar property, but only the sign of ∂_i as $\mathcal{P}\imath^i$ is invariant: The change in sign of the pseudoscalar charge is cancelled by the inverted direction of flow of the magnetic current. In summary, it became clear that the Maxwell-equations show a transformation behaviour under $\mathcal{C}$, $\mathcal{P}$ and $\mathcal{T}$.

A.8 *Electrostatic potential*

Maxwell's equations clarify the relation between the field configuration and the distribution of the charges as sources of the fields. As such, they enable us to compute the field configuration from the source; in the easiest case this would be an electrostatic field around an electric point charge q. Using the Gauß-law in integral form

$$\int_V \mathrm{dV}\,\partial_i\mathrm{D}^i = \int_V \mathrm{dV}\,\epsilon^{ij}\partial_i\mathrm{E}_j = \int_{\partial V} \mathrm{dS}_i\,\epsilon^{ij}\mathrm{E}_j = 4\pi\epsilon r^2\mathrm{E} = 4\pi\int_V \mathrm{dV}\,\rho = 4\pi q \tag{A.31}$$

		$\mathcal{C}$	$\mathcal{P}$	$\mathcal{T}$	$\mathcal{CP}$	$\mathcal{CT}$	$\mathcal{PT}$	$\mathcal{CPT}$
spatial derivative	∂_i	+	−	+	−	+	−	−
time derivative	∂_{ct}	+	+	−	+	−	−	−
electric charge density	ρ	−	+	+	−	−	+	−
electric current density	$\jmath^i$	−	−	−	+	+	+	−
magnetic charge density	τ	−	−	+	+	−	−	+
magnetic current density	$\imath^i$	−	+	−	−	+	−	+
dielectric displacement	D^i	−	−	+	+	−	−	+
electric field	E_i	−	−	+	+	−	−	+
magnetic induction	H_i	−	+	−	−	+	−	+
magnetic field	B^i	−	+	−	−	+	−	+

Table 1: Summary of the behaviour of all fields and sources in extended electrodynamics with electric and magnetic sources.

imposing spherical symmetry and working with an isotropic medium with dielectric constant ϵ (which implies $\epsilon^{ij} = \epsilon\gamma^{ij}$) leads to a radial field

$$E = \frac{q}{\epsilon\, r^2} \tag{A.32}$$

Clearly, the $1/r^2$-behaviour is a consequence of the growth of the surface area of spheres with increasing radius r, because the electric flux ϕ through every spherical shell is conserved. Positioning the charge q_1 at the position $\boldsymbol{r}_1$ and observing the field $\mathbf{E}$ at the position $\boldsymbol{r}$ would yield

$$E_i(\boldsymbol{r}) = \frac{q_1}{|\boldsymbol{r}-\boldsymbol{r}_1|^2}\,\gamma_{ij}\frac{(\boldsymbol{r}-\boldsymbol{r}_1)^j}{|\boldsymbol{r}-\boldsymbol{r}_1|} = q_1\,\gamma_{ij}\frac{(\boldsymbol{r}-\boldsymbol{r}_1)^j}{|\boldsymbol{r}-\boldsymbol{r}_1|^3} \tag{A.33}$$

where $\boldsymbol{r}-\boldsymbol{r}_1/|\boldsymbol{r}-\boldsymbol{r}_1|$ is a unit vector pointing from the charge q_1 to the observation point, converted with γ_{ij} into unit linear form. For a test charge, positive by convention, this would then yield a repulsive force for positive q_1 and an attractive force for negative q_1. The electric field of a collection of N charges q_n, $n = 1\ldots N$ follows by superposition, as the Maxwell-equations are linear:

$$E_i(\boldsymbol{r}) = \sum_{n=1}^{N} q_n\,\gamma_{ij}\frac{(\boldsymbol{r}-\boldsymbol{r}_n)^j}{|\boldsymbol{r}-\boldsymbol{r}_n|^3} \tag{A.34}$$

Transitioning to the continuum limit and replacing the discrete charges q_n at positions $\boldsymbol{r}_n$ with a continuous charge density $\rho(\boldsymbol{r})$ requires to replace summations by volume integrals

$$q = \sum_{n=1}^{N} q_n = \int_V dV'\,\rho(\boldsymbol{r}') \tag{A.35}$$

such that the total charge q in the system is respected. Similar relations should hold for any weighted integral and weighted sum, such that

$$E_i(\boldsymbol{r}) = \sum_{n=1}^{N} q_n\,\gamma_{ij}\frac{(\boldsymbol{r}-\boldsymbol{r}_n)^j}{|\boldsymbol{r}-\boldsymbol{r}_n|^3} = \int_V dV'\,\rho(\boldsymbol{r}')\,\gamma_{ij}\frac{(\boldsymbol{r}-\boldsymbol{r}')^j}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \tag{A.36}$$

i.e. the electric field results by convolution of the charge density ρ with a vectorial integration kernel. An explicit calculation shows that

$$-\partial_i\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \gamma_{ij}\frac{(\boldsymbol{r}-\boldsymbol{r}')^j}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = +\partial_i'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{A.37}$$

with ∂_i acting on $\boldsymbol{r}$ and ∂_i' acting on $\boldsymbol{r}'$. Then,

$$E_i(\boldsymbol{r}) = \int_V dV'\,\rho(\boldsymbol{r}')\,\gamma_{ij}\frac{(\boldsymbol{r}-\boldsymbol{r}')^j}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = -\int_V dV'\,\rho(\boldsymbol{r}')\partial_i\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \\ -\partial_i\int_V dV'\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = -\partial_i\Phi(\boldsymbol{r}) \tag{A.38}$$

with the electrostatic potential Φ,

$$\Phi(\boldsymbol{r}) = \int_V dV'\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{A.39}$$

which is clearly given as the superposition of the Coulomb-potentials of each element of the charge distribution ρ. Mathematically, this relation would be interpreted as a convolution of the distribution of sources $\rho(\boldsymbol{r})$ with a kernel $1/r$, which is in this context called the Green-function.

Here, the electric field E_i is a gradient field, $E_i = -\partial_i\Phi$, such that

$$\epsilon^{ijk}\partial_j E_k = -\epsilon^{ijk}\partial_j\partial_k\Phi = 0 \tag{A.40}$$

consistent with the induction law $\epsilon^{ijk}\partial_j E_k = -\partial_{ct}B^i = 0$ which yields a vanishing result in the static case. The gradient field E_i is rotationless because the contraction of the antisymmetric ϵ^{ijk} with the symmetric $\partial_j\partial_k$ is necessarily zero.

There is a clear interpretation of the electrostatic potential as the energy needed to displace a test charge in the electric field

$$W = -q\int_A^B dr^i\,E_i = q\int_A^B dr^i\,\partial_i\Phi = q\int_A^B d\Phi = q\,(\Phi(\boldsymbol{r}_B) - \Phi(\boldsymbol{r}_A)) \tag{A.41}$$

as the field is conservative. The integrand $dr^i\partial_i\Phi = d\Phi$ should be interpreted as the gradient $\partial_i\Phi$ projected onto dr^i. Combining the relation $E_i = -\partial_i\Phi$ between the electric field and the potential with the Gauß-law $\partial_i D^i = \epsilon^{ij}\partial_i E_j = 4\pi\rho$ yields the Poisson-equation

$$\partial_i D^i = -\epsilon^{ij}\partial_i\partial_j\Phi = -\Delta\Phi = 4\pi\rho \quad\rightarrow\quad \Delta\Phi = -4\pi\rho \tag{A.42}$$

introducing the Laplace-operator, which assumes the general form $\Delta = \epsilon^{ij}\partial_i\partial_j$, falls back on $\Delta = \epsilon\gamma^{ij}\partial_i\partial_j$ in an isotropic medium and ultimately on $\Delta = \gamma^{ij}\partial_i\partial_j$ in vacuum.

It might be a fun thought to use the positive definiteness of the metric ϵ^{ij} to carry out a Cholesky-decomposition to $\epsilon^{ij} = e_m{}^i\gamma^{mn}e_n{}^j$ such that the Laplace-operator becomes $\Delta = \epsilon^{ij}\partial_i\partial_j = e_m{}^i\gamma^{mn}e_n{}^j\,\partial_i\partial_j = \gamma^{mn}\,e_m{}^i\partial_i e_n{}^j\partial_j$. Then, a coordinate transform with $x^i \rightarrow e^m{}_i x^i$ leads to $\partial_i \rightarrow e_m{}^i\partial_i$, such that in the new coordinates the Laplace operator takes on the Euclidean form, $\gamma^{mn}\partial_m\partial_n = \Delta$. In essence, the effect of an anisotropic medium can be absorbed by a (linear) change in coordinates. In the particular case of an isotropic medium, this amounts to a mere rescaling or to the usage of a different unit of length or charge, as the two are degenerate in Coulomb's law: The situation of an electric field of a charge inside a medium can be mapped onto a different charge in vacuum

A bit more surprising might be the realisation that the equipotential surfaces of the electric field around a point charge in an anisotropic medium would be ellipsoids, but in the change of coordinate suggested by the Cholesky-decomposition of ϵ^{ij} would become perfect spheres!

⚠ The Cholesky-decomposition, valid for any symmetric, positive definite matrix, is given by $e^i{}_m\gamma^{mn}e^j{}_n$ for the inverse permissivity ϵ^{ij}, and by $e_i{}^m\gamma_{mn}e_j{}^n$ for the permissivity ϵ_{ij}

The Laplace-operator in the Poisson-equation can be used to localise charges: Evaluating $-\partial_i\Phi$ at any position it determines the electric field E_i, whose divergence $\epsilon^{ij}\partial_i E_j = \partial_i D^i$ must, therefore, reflect the amount of charge $4\pi\rho$ at that point, in accordance with the Gauß-law. This operation is rather straightforward in the discrete case of point charges:

$$\Phi(\boldsymbol{r}) = \sum_{i=1}^{n}\frac{q_i}{|\boldsymbol{r}-\boldsymbol{r}_i|} \quad\rightarrow\quad \Delta\Phi = \sum_{i=1}^{n} q_i\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_i|} \propto -4\pi q_i \quad \text{if } \boldsymbol{r} = \boldsymbol{r}_i \tag{A.43}$$

and $\Delta\Phi = 0$ at any other position $\boldsymbol{r} \neq \boldsymbol{r}_i$. A direct calculation shows that this is in fact the case: Assuming spherical coordinates and positioning the charge at the origin implies indeed

$$\Delta\Phi = \Delta\frac{1}{r} = \frac{1}{r}\partial_r^2\left(r\frac{1}{r}\right) = 0 \tag{A.44}$$

for $r \neq 0$, but the expression can not be directly evaluated at the origin, as $1/r$ diverges. Instead, one can resort to averaging $\Delta\Phi$ over a small but finite integration volume V containing the charge and applying the Gauß-theorem:

$$\frac{1}{V}\int_V \mathrm{d}V\,\Delta\Phi = \frac{1}{V}\int_V \mathrm{d}V\,\epsilon^{ij}\partial_i\partial_j\Phi = \frac{1}{V}\int_{\partial V}\mathrm{d}S_i\,\underbrace{\epsilon^{ij}\partial_j\Phi}_{=D^i\propto 1/r^2} = -\frac{1}{V}\int_{\partial V}\underbrace{r^2\mathrm{d}\Omega}_{=\mathrm{d}S}\,\frac{1}{r^2} = -\frac{4\pi}{V}, \tag{A.45}$$

rewriting Δ as $\epsilon^{ij}\partial_i\partial_j$, making use of spherical symmetry and using $\nabla(1/r) = \partial_r(1/r) = -1/r^2$. This implies that the Laplace-operator Δ applied to $1/r$ yields either zero (if there is no charge at the location at which $\Delta\Phi$ is evaluated) or diverges (in the limit of $V \rightarrow 0$ if one has caught the charge in the integration volume). These two results can be summarised using Dirac's δ_D-function

$$\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -4\pi\delta_D(\boldsymbol{r}-\boldsymbol{r}'). \tag{A.46}$$

But electrodynamics is a continuum theory, and the computation has to work out for a charge density ρ as well: Integrating ρ over a volume V has to be the total charge q contained within that volume, such that the definitions

$$\rho(\boldsymbol{r}) = q\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}') \quad \rightarrow \quad \int_\mathrm{V} \mathrm{dV}\, \rho(\boldsymbol{r}) = q\int_\mathrm{V} \mathrm{dV}\, \delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}') = q \tag{A.47}$$

become consistent due to the normalisation of the δ_D-function. Then,

$$\Phi(\boldsymbol{r}) = \int_\mathrm{V} \mathrm{dV}'\, \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{A.48}$$

is the solution for the potential, as it solves the Poisson-equation

$$\Delta\Phi(\boldsymbol{r}) = \int_\mathrm{V} \mathrm{dV}'\, \rho(\boldsymbol{r}')\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \int_\mathrm{V} \mathrm{dV}'\, \rho(\boldsymbol{r}')(-4\pi)\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}') = -4\pi\rho(\boldsymbol{r}) \tag{A.49}$$

because of the shifting property of the δ_D-function

$$\int \mathrm{dV}'\, \rho(\boldsymbol{r}')\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}') = \rho(\boldsymbol{r}) \tag{A.50}$$

A collection of discrete point charges can be written as a charge density

$$\rho(\boldsymbol{r}) = \sum_{i=1}^{n} q_i\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}_i) \tag{A.51}$$

as a generalisation of equation (A.47), because

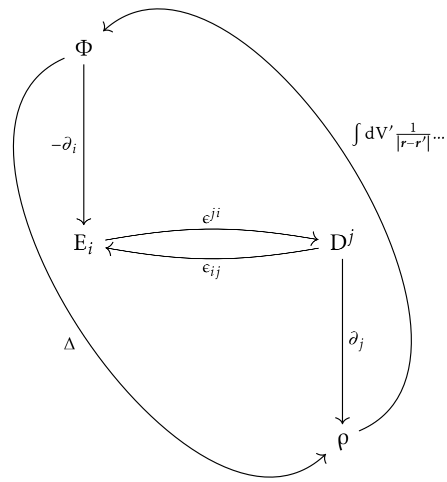
$$\Delta\Phi(\boldsymbol{r}) = \sum_{i=1}^{n} q_i\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_i|} = \sum_{i=1}^{n} q_i(-4\pi)\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}_i) = -4\pi\rho(\boldsymbol{r}) \tag{A.52}$$

making the concept of discrete point charges and a continuous charge distribution compatible. Unitwise, the δ_D-function is an inverse volume, because it is normalised to unity, $\int \mathrm{dV}\, \delta_\mathrm{D}(\boldsymbol{r}) = 1$, such that $q_i\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}_i)$ becomes the charge density $\rho(\boldsymbol{r})$.

Clearly, both Φ and $\mathrm{E}_i = -\partial_i\Phi$ can exist at points where ρ vanishes, but at these positions, the divergence $\epsilon^{ij}\partial_i\mathrm{E}_j$ and consequently $\Delta\Phi$ are necessarily zero. From this point of view one could argue that the tensor $\partial_i\partial_j\Phi$ would naturally decompose into a traceless part and a trace,

$$\partial_i\partial_j\Phi = \left(\partial_i\partial_j\Phi - \frac{\gamma_{ij}}{3}\Delta\Phi\right) + \frac{\gamma_{ij}}{3}\Delta\Phi \tag{A.53}$$

where the trace $\Delta\Phi$ reflects the contribution to the field generated by the charge density at the same point where $\Delta\Phi$ is evaluated, whereas the traceless part is the contribution to the electrical field sourced elsewhere. The relationships between source ρ, potential Φ and the fields E_i and D^i is summarised concisely in this diagram:



(A.54)

A.9 *Potential energy of a static charge distribution*

The potential Φ is in the *electrostatic* case related to the energy needed to displace a charge in the electric field. If one assembles a charge distribution, one would need to invest energy for doing so, as all charges would need to be moved from infinity (where the potential vanishes) to their dedicated positions. Let's do this step by step: The first charge q_1 is located at $\boldsymbol{r}_1$ and generates a potential Φ_1 at position $\boldsymbol{r}$

$$\Phi_1(\boldsymbol{r}) = \frac{q_1}{|\boldsymbol{r} - \boldsymbol{r}_1|} \tag{A.55}$$

Then, moving q_2 from infinity to $\boldsymbol{r}_2$ in the electric field that is already generated by q_1 requires the energy $W_2 = q_2\Phi_1(\boldsymbol{r}_2)$, and continuing with a third charge q_3 to be taken from infinity to $\boldsymbol{r}_3$ requires $W_3 = q_3\,(\Phi_1(\boldsymbol{r}_3) + \Phi_2(\boldsymbol{r}_3))$, which generalises to

$$W_n = q_n \sum_{m=1}^{n-1} \Phi_m(\boldsymbol{r}_n) \tag{A.56}$$

Adding up the amounts of work W_n needed for assembling the charge distribution suggests for the total energy W

$$W_{\mathrm{el}} = \sum_{n=1}^{N} W_n = \sum_{n=1}^{N} q_n \sum_{m=1}^{n-1} \Phi_m(\boldsymbol{r}_n) = \sum_{n=1}^{N} q_n \sum_{m=1}^{n-1} \frac{q_m}{|\boldsymbol{r}_n - \boldsymbol{r}_m|} = \frac{1}{2} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{q_m q_n}{|\boldsymbol{r}_n - \boldsymbol{r}_m|} \tag{A.57}$$

with a correction factor 1/2 due to the double counting, where we implicitly avoid the case $n = m$. In the continuum limit the relation becomes

$$W_{\mathrm{el}} = \frac{1}{2}\int_V \mathrm{d}V \int_V \mathrm{d}V' \,\frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{2}\int_V \mathrm{d}V\,\rho(\boldsymbol{r})\Phi(\boldsymbol{r}) = -\frac{1}{8\pi}\int_V \mathrm{d}V\,\Phi(\boldsymbol{r})\Delta\Phi(\boldsymbol{r}) \quad \text{(A.58)}$$

inserting the definition of the potential in the first and the Poisson equation in the second step, replacing ρ by ΔΦ. By making use of the Leibnitz-rule one can rewrite

$$\Phi\Delta\Phi = \Phi\epsilon^{ij}\partial_i\partial_j\Phi = \epsilon^{ij}\partial_i(\Phi\partial_j\Phi) - \epsilon^{ij}\partial_i\Phi\cdot\partial_j\Phi \quad \text{(A.59)}$$

and arrive at the reformulation by virtue of the Gauß-theorem,

$$W_{\mathrm{el}} = -\frac{1}{8\pi}\int_V \mathrm{d}V\,\epsilon^{ij}\partial_i(\Phi\partial_j\Phi) + \frac{1}{8\pi}\int_V \mathrm{d}V\,\epsilon^{ij}\partial_i\Phi\cdot\partial_j\Phi =$$

$$-\frac{1}{8\pi}\int_{\partial V} \mathrm{d}S_i\,\epsilon^{ij}(\Phi\partial_j\Phi) + \frac{1}{8\pi}\int_V \mathrm{d}V\,\epsilon^{ij}\partial_i\Phi\cdot\partial_j\Phi = \frac{1}{8\pi}\int_V \mathrm{d}V\,\epsilon^{ij}E_iE_j \quad \text{(A.60)}$$

where the first term typically vanishes faster than the surface area ∂V increases, as $\Phi \propto 1/r$ and $\partial\Phi \propto 1/r^2$ at large distances from the charge distribution, dominating over the increase of $\partial V \propto r^2$. This result implies that the electric field can be assigned an energy density

$$w_{\mathrm{el}} = \frac{\epsilon^{ij}E_iE_j}{8\pi} = \frac{E_iD^i}{8\pi} = \frac{\epsilon_{ij}D^iD^j}{8\pi} \quad \text{with} \quad W_{\mathrm{el}} = \int_V \mathrm{d}V\,w_{\mathrm{el}}, \quad \text{(A.61)}$$

where the dielectric tensor now acts as a metric for computing the energy density from the fields, making it invariant under transformations. Positive definiteness of ϵ_{ij} (and consequently, of ϵ^{ij}) ensures that the energy density for electric fields comes out as positive. The energy density associated with magnetic fields is given in complete analogy by

$$w_{\mathrm{mag}} = \frac{\mu^{ij}H_iH_j}{8\pi} = \frac{H_iB^i}{8\pi} = \frac{\mu_{ij}B^iB^j}{8\pi} \quad \text{with} \quad W_{\mathrm{mag}} = \int_V \mathrm{d}V\,w_{\mathrm{mag}}. \quad \text{(A.62)}$$

A.10 *Boundary conditions for fields on surfaces*

Maxwell's equations allow a direct statement about the behaviour of the electric fields at boundaries in the static case: If a surface carries a charge surface density σ, an application of the Gauß-theorem to a small volume V situated on the surface yields

$$\int_V \mathrm{d}V\,\partial_i D^i = \int_{\partial V} \mathrm{d}S_i\,D^i = 4\pi\int_{\partial V} \mathrm{d}S\,\sigma = 4\pi\sigma\Delta S = \Delta S\,(D_2^{\perp} - D_1^{\perp}) \quad \rightarrow \quad D_2^{\perp} = D_1^{\perp} + 4\pi\sigma \quad \text{(A.63)}$$

if the height of the integration volume is neglected; effectively one deals with a very flat box. Similarly, because $\epsilon^{ijk}\partial_jE_k = -\epsilon^{ijk}\partial_j\partial_k\Phi = 0$ for electrostatic fields,

$$\int_S \mathrm{d}S_i\, \epsilon^{ijk}\partial_j E_k = \int_{\partial S} \mathrm{d}r^i\, E_i = \Delta r\, (E_2^{\parallel} - E_1^{\parallel}) = 0 \quad \rightarrow \quad E_2^{\parallel} = E_1^{\parallel} \tag{A.64}$$

as an application of the Stokes-theorem to a small and flat area S perpendicular to the surface. The constitutive relations $D^i = \epsilon^{ij} E_j$ and its inverse then allow the computation of $E^{\perp}$ and $D^{\parallel}$.

B POTENTIAL THEORY

B.1 *Potential theory*

Computing the field configuration $E_i(\boldsymbol{r})$ for a given distribution of electric charges $\rho(\boldsymbol{r})$ in the case of electrostatics requires the solution of the Poisson-equation through a convolution integral

$$\Delta\Phi(\boldsymbol{r}) = -4\pi\rho(\boldsymbol{r}) \quad \rightarrow \quad \Phi(\boldsymbol{r}) = \int_V \mathrm{d}V'\, \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{B.65}$$

with subsequently determining the gradient $E_i(\boldsymbol{r}) = -\partial_i\Phi(\boldsymbol{r})$. The reason for taking the detour over the potential Φ is that Poisson-problems of the form

$$\Delta\Phi(\boldsymbol{r}) = -4\pi\rho(\boldsymbol{r}) \tag{B.66}$$

are scalar and very well understood, with a plethora of solution methods. They map a single scalar source ρ onto a scalar field Φ, and from this perspective it is clear that the components of $E_i = -\partial_i\Phi$ can not be independent from each other, as they have to have a vanishing rotation, $\epsilon^{ijk}\partial_j E_k = 0$ in the static case. Please note that the inverse operation, i.e. determining the charge density ρ at a given position from the potential is straightforward: It suffices to compute the divergence of the gradient of the electric potential, $\Delta\Phi = \epsilon^{ij}\partial_i\partial_j\Phi$ to obtain ρ up to a factor of -4π.

Essentially, one needs to worry about three issues: (i) the inversion of the differential operator Δ for isolating Φ, which is achieved with the Green-function method, (ii) dealing with a possibly complicated geometry of the charge distribution ρ, and (iii) including boundary conditions typical for elliptical partial differential equations such as the Poisson-equation. The second issue is less severe and almost automatically taken care of if the first and third issue are solved: As the Poisson-equation is linear, the potential of an entire charge distribution should result from the superposition of the potentials generated by each infinitesimal element of charge.

B.2 *Systematic construction of Green-functions*

Formally, the solution to the Poisson-equation can be thought of as applying an inverse operator Δ^{-1} for isolating Φ from the relation $\Delta\Phi = -4\pi\rho$: The well-known convolution integral

$$\Phi = \int_V \mathrm{d}V'\, \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{B.67}$$

provides a solution to the Poisson-equation and therefore, consistency requires

$$\Phi = \Delta^{-1}\Delta\Phi = -4\pi\Delta^{-1}\rho. \tag{B.68}$$

In this sense, the convolution

$$\int_V \mathrm{d}V' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \ldots \quad \text{is an inverse operation to} \quad \Delta\,[\ldots], \tag{B.69}$$

perfectly encapsulated in the relation

$$\Delta\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = -4\pi\delta_D(\boldsymbol{r}-\boldsymbol{r}'). \tag{B.70}$$

In this context, the integration kernel $1/|\boldsymbol{r}-\boldsymbol{r}'|$ is called the Green-function of the differential operator Δ (in three dimensions), and corresponds to the potential of a unit point charge. Although it is bad style (in my opinion), the notation Δ^{-1} can be used for denoting the convolution eqn. (B.69), and one formally solves the Poisson-equation by application of the Δ^{-1}-operator, $\Phi = \Delta^{-1}\Delta\Phi = -4\pi\,\Delta^{-1}\rho$, through convolution.

Up to this point, the approach was very intuitive: The Gauß-law suggests that the electrostatic field around a point charge should be $\propto 1/r^2$ and conservative, such that a potential exists. The potential has to have a scaling $\propto 1/r$ for its gradient to describe the electric field. But there should be a general way of constructing the Green-function Δ^{-1} for any differential operator Δ. For that purpose, one introduces the Fourier-transform of the potential

$$\Phi(\boldsymbol{k}) = \int_V \mathrm{d}V\,\Phi(\boldsymbol{r})\exp(-\mathrm{i}k_i r^i) \quad\leftrightarrow\quad \Phi(\boldsymbol{r}) = \int_V \frac{\mathrm{d}^3k}{(2\pi)^3}\,\Phi(\boldsymbol{k})\exp(+\mathrm{i}k_i r^i) \tag{B.71}$$

as well as of the charge density

$$\rho(\boldsymbol{k}) = \int_V \mathrm{d}V\,\rho(\boldsymbol{r})\exp(-\mathrm{i}k_i r^i) \quad\leftrightarrow\quad \rho(\boldsymbol{r}) = \int_V \frac{\mathrm{d}^3k}{(2\pi)^3}\,\rho(\boldsymbol{k})\exp(+\mathrm{i}k_i r^i) \tag{B.72}$$

Then, the Poisson-equation becomes

$$\Delta\Phi(\boldsymbol{r}) = \Delta\int_V \frac{\mathrm{d}^3k}{(2\pi)^3}\,\Phi(\boldsymbol{k})\exp(+\mathrm{i}k_i r^i) = \int_V \frac{\mathrm{d}^3k}{(2\pi)^3}\,\Phi(\boldsymbol{k})\Delta\exp(+\mathrm{i}k_i r^i) =$$
$$\int_V \frac{\mathrm{d}^3k}{(2\pi)^3}\,\Phi(\boldsymbol{k})(-\gamma^{ab}k_a k_b)\exp(+\mathrm{i}k_i r^i) = -4\pi\int_V \frac{\mathrm{d}^3k}{(2\pi)^3}\,\rho(\boldsymbol{k})\exp(+\mathrm{i}k_i r^i) = -4\pi\rho(\boldsymbol{r}) \tag{B.73}$$

as $\Delta = \gamma^{ab}\partial_a\partial_b$ acts on the plane wave $\exp(+\mathrm{i}k_i r^i)$ twice and generates a pre-factor $-\gamma^{ab}k_a k_b = -k^2$, for an isotropic medium for simplicity. Comparing the two Fourier-transforms suggests that

$$\Delta\Phi(\boldsymbol{r}) = -4\pi\rho(\boldsymbol{r}) \quad\rightarrow\quad k^2\Phi(\boldsymbol{k}) = 4\pi\rho(\boldsymbol{k}), \quad \text{solved by} \quad \Phi(\boldsymbol{k}) = \frac{4\pi}{k^2}\rho(\boldsymbol{k}) \tag{B.74}$$

Most interestingly, the (partial) differential equation has become a straightforward algebraic equation, which is readily solvable. Clearly, one can isolate Φ through division by $-k^2$ in Fourier-space, as illustrated by the diagram:

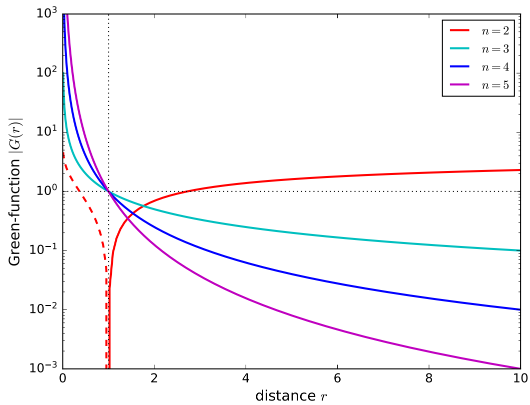


Figure 2: Green-functions G(r) *of the Laplace-operator* Δ *in different dimensions* n*.*

$$\begin{array}{ccc} \Phi(\boldsymbol{r}) & \xleftarrow{\mathcal{F}^{-1}} & \Phi(\boldsymbol{k}) \\ \Delta \big\downarrow & & \big\downarrow -\gamma^{ij}k_ik_j \\ \rho(\boldsymbol{r}) & \xrightarrow{\mathcal{F}} & \rho(\boldsymbol{k}) \end{array} \tag{B.75}$$

which suggests that $\Phi = \mathcal{F}^{-1}\left(4\pi/k^2\,\mathcal{F}(\rho)\right)$, as the complication of solving the Poisson-equation is replaced by finding the Fourier-transform and its inverse. There are even performance advantages of taking the detour through Fourier-space, as there are very powerful and efficient Fourier-transform algorithms.

In fact, multiplications in Fourier-space are convolutions in real space, which implies for our case that the product between the Fourier-transformed Green-function $4\pi/k^2$ and the Fourier-transformed charge distribution $\rho(\boldsymbol{k}) = \mathcal{F}(\rho)$ yields the Fourier-transformed potential $\Phi(\boldsymbol{k})$ in this detour. At the same time, the Fourier-transform of $4\pi/k^2$ must be equal to $1/r$, which we already know to be the Green-function for Δ in 3 dimensions.

Let's repeat this construction for the Laplace-operator and derive an expression for the Green-function which is generalisable beyond $n = 3$ dimensions: In general, the Green-function $\mathrm{G}(\boldsymbol{r}-\boldsymbol{r}')$ is defined as the potential for a unit point charge element, represented by a Dirac-δ_D, so the Poisson-equation needs to be fulfilled:

$$\Delta\mathrm{G}(\boldsymbol{r}-\boldsymbol{r}') = -4\pi\delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}') \tag{B.76}$$

Both the Green-function as well as the Dirac-δ_D have a Fourier representation:

$$\mathrm{G}(\boldsymbol{r}-\boldsymbol{r}') = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\mathrm{G}(\boldsymbol{k})\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i) \quad\text{and}\quad \delta_\mathrm{D}(\boldsymbol{r}-\boldsymbol{r}') = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i), \tag{B.77}$$

where δ_D has a constant amplitude in Fourier-space. Substituting into the Poisson-equation yields

$$\Delta G(\boldsymbol{r}-\boldsymbol{r}') = \Delta \int \frac{d^3k}{(2\pi)^3}\, G(\boldsymbol{k}) \exp(ik_i(\boldsymbol{r}-\boldsymbol{r}')^i) = \int \frac{d^3k}{(2\pi)^3}\,(-k^2) G(\boldsymbol{k}) \exp(ik_i(\boldsymbol{r}-\boldsymbol{r}')^i) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, \exp(ik_i(\boldsymbol{r}-\boldsymbol{r}')^i) = -4\pi\delta_D(\boldsymbol{r}-\boldsymbol{r}') \quad \text{(B.78)}$$

such that

$$G(\boldsymbol{k}) = \frac{4\pi}{k^2} \quad \text{(B.79)}$$

because each differentiation ∂_i generates a prefactor of ik_i. While the proportionality $\propto 1/k^2$ is valid in any number of dimensions, transforming back according to

$$G(\boldsymbol{r}-\boldsymbol{r}') = \int \frac{d^nk}{(2\pi)^n}\, G(\boldsymbol{k}) \exp(ik_i(\boldsymbol{r}-\boldsymbol{r}')^i) \quad \text{(B.80)}$$

leads to different results due to volume element $d^nk \propto k^{n-1}dk$ depending on dimensionality. In addition, 4π is just the full solid angle in three dimensions, and would need to be changed if the dimensionality is different.

B.3 *Green-theorems*

For showing the uniqueness of solutions to potential problems and for incorporating boundary conditions one needs the two Green-theorems, which are readily derived as particular cases of the Gauß-theorem. Defining

$$A_i(\boldsymbol{r}') = \phi(\boldsymbol{r}')\partial_i'\psi(\boldsymbol{r}') \quad \text{with two scalar fields } \phi,\ \psi \quad \text{(B.81)}$$

that all depend on the primed coordinate for convenience in the derivations later on, gives

$$\gamma^{ij}\partial_i' A_j = \gamma^{ij}\partial_i'\left(\phi\partial_j'\psi\right) = \gamma^{ij}\partial_i'\phi\,\partial_j'\psi + \phi\Delta'\psi \quad \text{(B.82)}$$

due to the Leibnitz-rule, and by writing $\gamma^{ij}\partial_i'\partial_j' = \Delta'$. Applying the Gauss-theorem yields the first Green-theorem

$$\int_V dV'\,\gamma^{ij}\partial_i' A_j = \int_V dV'\,\gamma^{ij}\partial_i'\left(\phi\partial_j'\psi\right) = \int_V dV'\left(\gamma^{ij}\partial_i'\phi\,\partial_j'\psi + \phi\Delta'\psi\right) = \int_{\partial V} dS_i'\,\gamma^{ij}A_j = \int_{\partial V} dS_i'\,\gamma^{ij}\left(\phi\partial_j'\psi\right) \quad \text{(B.83)}$$

The right side of the first Green-theorem,

$$\int_V dV'\,\gamma^{ij}\partial_i'\left(\phi\partial_j'\psi\right) = \int_{\partial V} dS_i'\,\gamma^{ij}\left(\phi\partial_j'\psi\right), \quad \text{(B.84)}$$

can be interpreted as the scalar product of $\phi\partial'_j\psi$ with the surface normal dS'_i of the area element.

The second Green-theorem is obtained by interchanging the fields $\phi \leftrightarrow \psi$ in the first Green-theorem and by subtracting both expressions:

$$\int_V dV' \left(\phi\gamma^{ij}\partial'_i\partial'_j\psi - \psi\gamma^{ij}\partial'_i\partial'_j\phi\right) = \int_{\partial V} dS'_i\ \gamma^{ij}\left(\phi\partial'_j\psi - \psi\partial'_j\phi\right) \tag{B.85}$$

as the symmetric mixed term $\gamma^{ij}\partial'_i\psi\,\partial'_j\phi = \gamma^{ij}\partial'_i\phi\,\partial'_j\psi$ cancels.

The potential of an electrostatic problem is unique: For a given ρ there can be only a single potential Φ, defined up to an at most additive constant, which can be proved by contradiction. If there were two solutions

$$\Delta\Phi_1 = -4\pi\rho \quad \text{as well as} \quad \Delta\Phi_2 = -4\pi\rho \quad \rightarrow \quad \Delta(\Phi_1 - \Phi_2) = \Delta\delta = 0 \tag{B.86}$$

their difference $\delta = \Phi_1 - \Phi_2$ would fulfil the Laplace-equation $\Delta\delta = 0$, as shown by subtraction. Substituting δ into the first Green-theorem gives

$$\int_V dV' \left[\delta\gamma^{ij}\partial'_i\partial'_j\delta - \gamma^{ij}\partial'_i\delta\partial'_j\delta\right] = \int_{\partial V} dS'_i\ \gamma^{ij}\delta\partial'_j\delta = 0 \tag{B.87}$$

The surface-integral vanishes if proper boundary conditions are chosen on ∂V: Either, if $\Phi_1 = \Phi_2$ or $\delta = 0$ on ∂V is set (Dirichlet) or if $\partial'_j\Phi_1 = \partial'_j\Phi_2$ or $\partial'_j\delta = 0$ on ∂V (Neumann). With $\Delta'\delta = 0$ being zero because both Φ_1 and Φ_2 are solutions for the same source one arrives at

$$\int_V dV'\ \gamma^{ij}\delta\partial'_j\delta = 0 \tag{B.88}$$

which implies that $\partial^{ij}\partial'_i\delta\partial'_j\delta = 0$, as the integrand is positive definite and must vanish over any specified volume. As a consequence, $\Phi_2 = \Phi_1 + \text{const}$ at most, and the constant must vanish for Dirichlet-conditions because of $\delta = 0$ on ∂V.

The solution to the Poisson-equation did not yet incorporate boundary conditions like specified values on surfaces or specified gradients. Setting

$$\psi(\boldsymbol{r}') \equiv \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} \quad \rightarrow \quad \Delta'\psi(\boldsymbol{r}') = -4\pi\delta_D(\boldsymbol{r} - \boldsymbol{r}') \tag{B.89}$$

as well as

$$\phi(\boldsymbol{r}') \equiv \Phi(\boldsymbol{r}') \quad \rightarrow \quad \Delta'\phi(\boldsymbol{r}') = -4\pi\rho(\boldsymbol{r}') \tag{B.90}$$

suggests for the volume integrals

$$\int_V \mathrm{d}V' \left(\phi\gamma^{ij}\partial'_i\partial'_j\psi - \psi\gamma^{ij}\partial'_i\partial'_j\phi\right) = \int_V \mathrm{d}V' \left(\Phi(\boldsymbol{r}')(-4\pi\delta_D(\boldsymbol{r}-\boldsymbol{r}')) + \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}4\pi\rho(\boldsymbol{r}')\right) = -4\pi\Phi(\boldsymbol{r}) + 4\pi\int_V \mathrm{d}V' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \quad \text{(B.91)}$$

and for the surface integrals

$$\int_{\partial V} \mathrm{d}S'_i\gamma^{ij}\left(\phi\partial'_i\psi - \psi\partial'_j\phi\right) = \int_{\partial V} \mathrm{d}S'_i\gamma^{ij}\left(\Phi(\boldsymbol{r}')\partial'_j\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\partial'_j\Phi(\boldsymbol{r}')\right) \quad \text{(B.92)}$$

Assembling the entire expression gives the relation

$$\Phi(\boldsymbol{r}) = \int_V \mathrm{d}V' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{1}{4\pi}\int_{\partial V} \mathrm{d}S'_i\,\gamma^{ij}\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\partial'_j\Phi(\boldsymbol{r}') - \Phi(\boldsymbol{r}')\partial'_j\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right) \quad \text{(B.93)}$$

with the volume integral reiterating the conventional way of computing Φ from ρ, augmented by two additional contributions, one representing Neumann-boundary conditions with $\nabla\Phi$ on ∂V and the second representing Dirichlet-boundary conditions with Φ on ∂V. If the boundary is at infinity, both $1/r\,\nabla\Phi$ and $\Phi\,\nabla 1/r$ tend to zero as $1/r^3$, so the first term is the only one to survive. Interestingly, the formula suggests that there can be a nontrivial potential Φ even though ρ might be zero: Then, the potential is determined by Φ and $\nabla\Phi$ on the boundary. It might be a surprisingly sensible question, if one can construct a charge distribution that replaces the boundary conditions in an otherwise unconstrained potential problem, and the question can be positively answered: Any potential Φ is linked to a distribution of sources ρ through the Poisson-equation $\Delta\Phi = -4\pi\rho$, so setting $\rho = -\Delta\Phi/(4\pi)$ would be consistent with a potential fulfilling the boundary conditions, which is exactly the method of mirror charges.

B.4 *Spherical multipole expansion*

The Green-function $1/|\boldsymbol{r}-\boldsymbol{r}'|$ is the correct convolution kernel for computing the potential Φ for any charge distribution ρ in fulfilment of the Poisson-equation $\Delta\Phi = -4\pi\rho$. But there might be cases where an approximate computation of Φ is sufficient, in particular because intuitively, any localised charge distribution should generate a Coulomb-like spherically symmetric $1/r$-potential at large distances, with deviations only appearing at smaller distances: This is shown in Fig. 3, where one of isopotential surfaces is given for a uniformly charged cube. With increasing distance (and correspondingly, lower values for Φ), the surfaces become more and more spherical, as expected from the Coulomb-potential of a point charge. The effect is more pronounced in Fig. 4, where the isocontours of the potential of a charge distribution with four equal charges in the corners of a tetrahedron is shown.

In fact, expanding the Green-function leads to

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{\sqrt{r^2 - 2rr'\mu + r'^2}} = \frac{1}{r}\frac{1}{\sqrt{1 - 2\frac{r'}{r}\mu + \frac{r'^2}{r^2}}} \tag{B.94}$$

where $\mu = \cos\theta$ is the cosine of the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$. If one assumes now that the observation point $\boldsymbol{r}$ is far away from the charge distribution (and $\boldsymbol{r}'$ points by definition of the convolution relation to every charge element), then $r \gg r'$ and the root can be expanded:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} \mathrm{P}_\ell(\mu) \tag{B.95}$$

where $\mathrm{P}_\ell(\mu)$ are the Legendre-polynomials. They follow explicitly from the relation

$$\frac{1}{\sqrt{1-2\mu x + x^2}} = \sum_{\ell=0}^{\infty} \mathrm{P}_\ell(\mu) x^\ell \tag{B.96}$$

by ℓ-fold differentiation with respect to $x = r'/r \ll 1$ and successive setting of $x = 0$. Explicitly, this would result in $\mathrm{P}_0(\mu) = 1$, $\mathrm{P}_1(\mu) = \mu$ and $\mathrm{P}_2(\mu) = (3\mu^2 - 1)/2$.

Now, one can bridge between the Legendre-polynomals $\mathrm{P}_\ell(\cos\gamma)$ and the spherical harmonics $\mathrm{Y}_{\ell m}(\theta,\varphi)$ with the addition theorem

$$\mathrm{P}_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{+\ell} \mathrm{Y}_{\ell m}(\theta,\varphi)\mathrm{Y}^*_{\ell m}(\theta',\varphi') \tag{B.97}$$

with γ being the angle between (θ,φ) and (θ',φ'). These spherical harmonics are waves on the surface of the sphere

$$\Delta \mathrm{Y}_{\ell m}(\theta,\varphi) = -\ell(\ell+1)\mathrm{Y}_{\ell m}(\theta,\varphi), \quad \text{analogous to} \quad \Delta\exp(\pm \mathrm{i}k_i r^i) = -\gamma^{ab}k_a k_b \exp(\pm \mathrm{i}k_i r^i) \tag{B.98}$$

so that ℓ plays the role of a wave number, and its inverse reflects the wave length (in radians) of the waves. The spherical harmonics (for details, see Sect. X.6) constitute therefore a harmonic system and are naturally related to Fourier-transforms, and generalise the idea of harmonic analysis to functions defined on the surface of a sphere.

Replacing the Legendre-polynomials by spherical harmonics leads to

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1}\frac{r'^\ell}{r^{\ell+1}} \mathrm{Y}_{\ell m}(\theta,\varphi)\mathrm{Y}^*_{\ell m}(\theta',\varphi') \tag{B.99}$$

which can be substituted into the expression of the potential

$$\Phi(\boldsymbol{r}) = \int_V \mathrm{d}V'\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\frac{4\pi}{2\ell+1}\frac{1}{r^{\ell+1}}\mathrm{Y}_{\ell m}(\theta,\varphi)\times\int_V \mathrm{d}V'\,\rho(\boldsymbol{r}')r'^\ell \mathrm{Y}^*_{\ell m}(\theta',\varphi'). \tag{B.100}$$

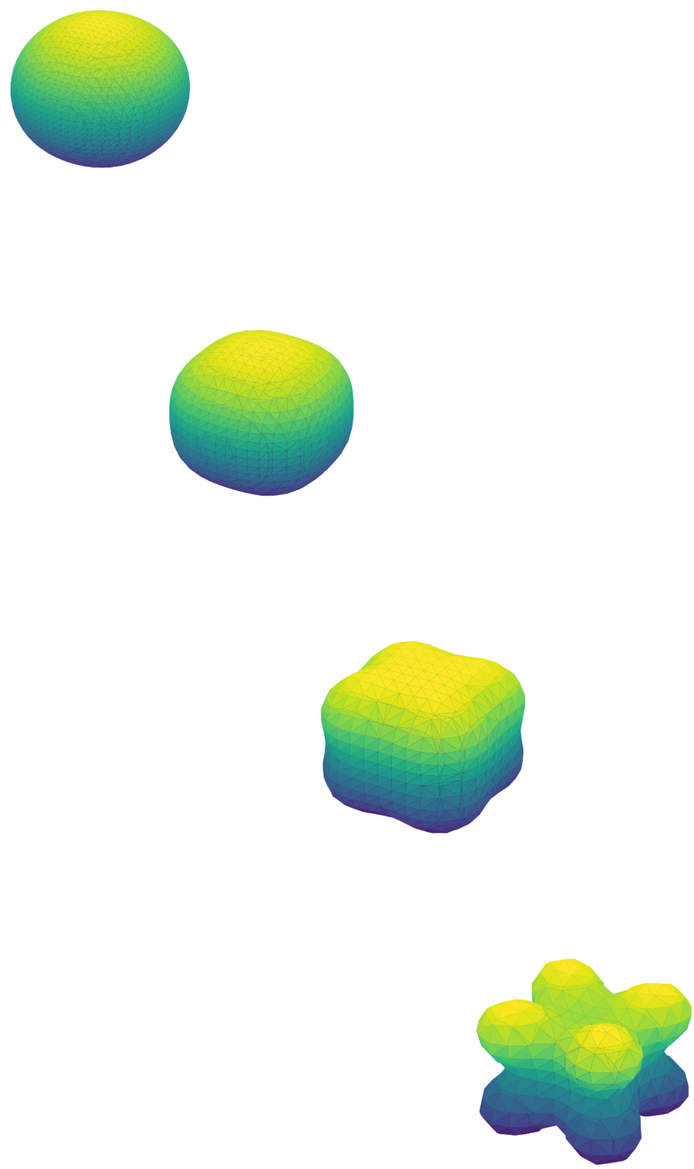


Figure 3: Isopotential surfaces of the potential sourced by eight equal charges, situated at the corners of a cube, at decreasing distance

Figure 4: Isopotential surfaces of the potential sourced by four equal charges, situated at the corners of an tetrahedron, at decreasing distance

This formula is remarkable because it separates properties of the charge distribution from the field that it would generate: Sorting the variables into primed and unprimed leads to the definition of multipole moments $q_{\ell m}$

$$q_{\ell m} = \int_V dV' \, \rho(\boldsymbol{r}') r'^{\ell} Y^*_{\ell m}(\theta', \varphi'). \tag{B.101}$$

The multipole moments are a complete characterisation of the charge distribution and contain information about the magnitude of the charge, the spatial size, the shape, asphericity and orientation. Each of the multipoles is an independent contribution to the potential Φ, whose influence decreases as $1/r^{\ell+1}$

$$\Phi(\boldsymbol{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1} \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi) q_{\ell m}, \tag{B.102}$$

which is amazingly practical, as higher-order multipoles generate contributions to Φ which decay faster and faster with increasing distance. At large distances, only the lowest order multipole can contribute, and it is sensible to expect that this contribution should be a spherically symmetric potential determined by the total charge. In fact, there is only $m = 0$ permissible for $\ell = 0$, so that there is a single coefficient q_{00},

$$q_{00} = \int_V dV' \underbrace{r'^0}_{=1} \rho(\boldsymbol{r}') \underbrace{Y^*_{00}(\theta', \varphi')}_{=\frac{1}{\sqrt{4\pi}}} = \frac{q}{\sqrt{4\pi}} \tag{B.103}$$

Therefore, the monopole q_{00} is the total charge of the system q, up to a factor of $1/\sqrt{4\pi}$. At large distances, this term would dominate the multipole expansion and generate a $1/r$-like contribution to the potential Φ, in agreement with intuition that the potential, viewed from a large distance of a somehow localised charge distribution, should have this form.

The dipole $\ell = 1$ allows the three choices $m = -1, 0, +1$, therefore, there are three dipole moments

$$q_{1m} = \int_V dV' \, r' \rho(\boldsymbol{r}') Y^*_{1m}(\theta', \varphi') \tag{B.104}$$

whose fundamental functional form is that of "charge × distance", and carrying the sequence further one defines 5 quadrupole moments q_{2m} for $m = -2\ell, -\ell, 0, +\ell, +2\ell$

$$q_{2m} = \int_V dV' \, r'^2 \rho(\boldsymbol{r}') Y^*_{2m}(\theta', \varphi') \tag{B.105}$$

with a fundamental scaling "charge × area", and it is obvious how this would generalise to higher order multipoles such as octupoles and hexadecupoles. The idea is always that the charge distribution is split up into coefficients $q_{\ell m}$ that by construction look for smaller and smaller structures and that are sensitive to the spatial extent (through the weighting with r'^{ℓ}) of the charge distribution, and to its asphericity and orientation (through projection onto the spherical harmonics $Y_{\ell m}(\theta', \varphi')$).

Formally, one needs the full set of multipole moments for writing down the multipole expansion, but there is a hermiticity constraint just as in the case of the Fourier-components of negative frequency for a real-valued function. Fundamentally, one has

$$Y_{\ell m}^{*}(\theta,\varphi) = (-1)^m Y_{\ell,-m}(\theta,\varphi) \tag{B.106}$$

which maps onto the relation

$$q_{\ell m}^{*} = \int_V \mathrm{d}V'\, r'^{\ell} \rho(\boldsymbol{r}') Y_{\ell m}(\theta',\varphi') = (-1)^m \int_V \mathrm{d}V'\, r'^{\ell} \rho(\boldsymbol{r}') Y_{\ell,-m}^{*}(\theta',\varphi') = (-1)^m q_{\ell,-m} \tag{B.107}$$

so that there are not $2\ell+1$ but rather only $\ell+1$ independent multipole coefficients for a real-valued charge distribution. With this realisation it is clear that the charged cube in Fig. 3 can only exhibit a monopole and an octupole at lowest order. While the monopole gives rise to a straightforward spherically symmetric Coulomb-potential, the octupole contribution falls of very quickly $\propto 1/r^4$, so that it only matters very close to the surface of the cube, and renders the isopotential surface non-spherical.

B.5 *Cartesian multipole expansion*

There is an alternative approach to multipole expansions in terms of Cartesian coordinates, where the Green-function of a charge distribution localised around the origin of the coordinate system is Taylor-expanded at $\boldsymbol{r}' = 0$ with respect to the variable $\boldsymbol{r}'$, while $\boldsymbol{r}$ is kept fixed:

$$G(\boldsymbol{r},\boldsymbol{r}') \simeq G\Big|_{\boldsymbol{r}'=0} + \partial_i' G\Big|_{\boldsymbol{r}'=0}(x')^i + \frac{1}{2!}\partial_i'\partial_j' G\Big|_{\boldsymbol{r}'=0}(x')^i(x')^j + \cdots \tag{B.108}$$

The necessary derivatives of $G(\boldsymbol{r},\boldsymbol{r}')$ at $\boldsymbol{r}' = 0$ are easily computed to be

$$G\Big|_{\boldsymbol{r}'=0} = \frac{1}{r},\quad \partial_i' G\Big|_{\boldsymbol{r}'=0} = \frac{\gamma_{ia}x^a}{r^3},\quad \text{and}\quad \partial_i'\partial_j' G\Big|_{\boldsymbol{r}'=0} = \frac{3\gamma_{ia}x^a\gamma_{jb}x^b - r^2\gamma_{ij}}{r^5} \tag{B.109}$$

using the explicit form of the Green-function in Cartesian coordinates

$$G(\boldsymbol{r},\boldsymbol{r}') = \left[\gamma_{ab}(\boldsymbol{r}-\boldsymbol{r}')^a(\boldsymbol{r}-\boldsymbol{r}')^b\right]^{-1/2}. \tag{B.110}$$

Here, we abbreviate $r^2 = \gamma_{ij}x^i x^j$ as the Euclidean norm of $\boldsymbol{r}$. Then, the potential Φ is given at $r \gg r'$ as

$$\Phi(\boldsymbol{r}) = \int_V \mathrm{d}V' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \simeq \int_V \mathrm{d}V' \rho(\boldsymbol{r}')\left[\frac{1}{r} + \frac{\gamma_{ia}x^a(x')^i}{r^3} + \frac{3\gamma_{ia}x^a\gamma_{jb}x^b - r^2\gamma_{ij}}{r^5}(x')^i(x')^j + \cdots\right] \tag{B.111}$$

Applying the integration to each term in the series while interchanging summation and integration gives

$$\Phi(\boldsymbol{r}) \simeq \frac{1}{r}\int_V dV'\ \rho(\boldsymbol{r}') + \frac{x^a}{r^3}\int_V dV'\ \rho(\boldsymbol{r}')\gamma_{ia}(x')^i + \frac{1}{2!}\frac{3\gamma_{ia}x^a\gamma_{jb}x^b - r^2\gamma_{ij}}{r^5}\int_V dV'\ \rho(\boldsymbol{r}')(x')^i(x')^j \quad \text{(B.112)}$$

where we can identify the Cartesian multipole moments: The total charge q in the first term, the dipole moment q_a in the second term, and the quadrupole moment in the last term. They contribute to the potential Φ with increasing powers of $1/r$, so that their influence at large distance decreases with multipole order.

There might be an aesthetic issue, as $3\gamma_{ai}x^a\gamma bjx^b - r^2\gamma ij$ is not mirrored in the primed coordinate in the quadrupole term, likewise one might be irritated why there seem to be six Cartesian multipole moments (There are 6 independent choices for i and j in the tensor $(x')^i(x')^j$) but only five in spherical coordinates (The index m can assume the 5 different values -2, -1, 0, 1 and 2 for $\ell = 2$). In order to remedy this issue, one adds a zero in the expression for the quadrupole moment

$$\frac{3\gamma_{ai}x^a\gamma_{jb}x^b - r^2\gamma_{ij}}{r^5}\int_V dV'\ \Big(\rho(\boldsymbol{r}')\ (x')^i(x')^j \underbrace{-r'^2\frac{\gamma^{ij}}{3} + \overbrace{r'^2\frac{\gamma^{ij}}{3}}^{\text{becomes zero}}}_{=0}\Big) \quad \text{(B.113)}$$

The last term in particular can be simplified, as in its contraction with the prefactor one can write:

$$\frac{3\gamma_{ai}x^a\gamma_{jb}x^b - r^2\gamma_{ij}}{r^5}\int_V dV'\ \rho(\boldsymbol{r}')r'^2\frac{\gamma^{ij}}{3} = 0 \quad \text{(B.114)}$$

because of $\gamma_{ai}\gamma_{bj}\gamma^{ij}x^ax^b = \gamma_{ab}x^ax^b = r^2$ and because $\gamma_{ij}\gamma^{ij} = \delta^i_i = 3$. Therefore, only the combination of the first two terms remain, explicitly

$$\frac{3\gamma_{ai}x^a\gamma_{bj}x^b - r^2\gamma_{ij}}{r^5}\int_V dV'\ \rho(\boldsymbol{r}')\left(3(x')^i(x')^j - r'^2\gamma^{ij}\right) = \frac{3x^ax^b - r^2\gamma^{ab}}{r^5}\int_V dV'\ \rho(\boldsymbol{r}')\left(3\gamma_{ai}(x')^a\gamma_{bj}(x')^b - r'^2\gamma^{ab}\right) \quad \text{(B.115)}$$

establishing an identical structure in the quadrupole term. Summarising all terms then yields the final result for the potential

$$\Phi(\boldsymbol{r}) = \frac{q}{r} + q_i\frac{x^i}{r^3} + q_{ij}\frac{3x^ix^j - r^2\gamma^{ij}}{r^5} \quad \text{(B.116)}$$

with the monopole that shows the expected $1/r$-behaviour, followed by the dipole term with a fundamental scaling $\propto 1/r^2$ and an angular cosine-like behaviour en-

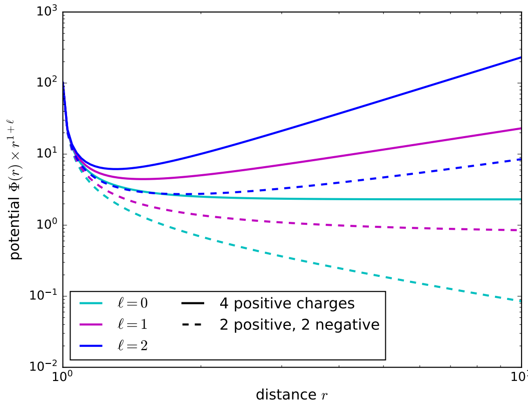


Figure 5: Potential Φ of the tetrahedron as a function of distance r, in the representation $\Phi \times r^{1+\ell}$, therefore, a flat section indicates a scaling $\Phi \propto 1/r^{1+\ell}$.

capsulated in the scalar product $q_i x^i$. The quadrupolar term decreases $\propto 1/r^3$ with distance. The moments read

$$q = \int\limits_V \mathrm{d}V'\ \rho(\boldsymbol{r}'), \quad q_i = \int\limits_V \mathrm{d}V'\ \rho(\boldsymbol{r}')\ \gamma_{ai}(x')^a, \tag{B.117}$$

and

$$q_{ij} = \int\limits_V \mathrm{d}V'\ \rho(\boldsymbol{r}') \left(\gamma_{ai}(x')^a \gamma_{bj}(x')^b - \frac{r'^2}{3}\gamma_{ij}\right) \tag{B.118}$$

The typical scaling of the potential Φ proportional to $1/r$ for the monopole, $1/r^2$ for the dipole and $1/r^3$ for the quadrupole is illustrated in Fig. 5 for the example of the tetrahedron. Any flat section of $\Phi \times r^{1+\ell}$ as a function of r indicates exactly the behaviour $\Phi \propto 1/r^{1+\ell}$. If there are four positive charges in the corners of the tetrahedron, one sees a dominating Coulomb-potential at large distances, while at shorter distances there is a dipole and a quadrupole contribution. If there are two positive and two negative charges, however, the total charge is zero and there can not be a Coulomb-type contribution to the potential: In fact there is no section with a flat $\Phi \times r$ as a function of r in this case. There is, however, a dominating dipole potential for large radii, and a quadrupolar contribution at small distances.

And additionally, this new definition of the quadrupole moment is traceless,

$$q^i{}_i = \gamma^{ij} q_{ij} = \int\limits_V \mathrm{d}V'\ \rho(\boldsymbol{r})\ \left(3\gamma_{ai}(x')^a \gamma_{jb}(x')^b - r'^2\gamma_{ij}\right)\gamma^{ij} = 0 \tag{B.119}$$

such that the Cartesian quadrupole moment q_{ij}, as a symmetric, traceless tensor in 3 dimensions has 5 instead of 6 degrees of freedom, commensurate with $q_{\ell m}$ for $\ell = 2$ in spherical coordinates.

B.6 *Potential energy of a charge distribution in a potential*

The same result, perhaps with a bit more physical insight, can be reached by considering the interaction between a charge distribution ρ and an external field Φ. The associated energy W is given by

$$W_{el} = \frac{1}{2}\int_V dV\, \rho(\boldsymbol{r})\Phi(\boldsymbol{r}) \tag{B.120}$$

where ρ acts now as a test charge distribution situated at $\boldsymbol{r} = 0$ in a potential Φ that gets Taylor-expanded around $\boldsymbol{r} = 0$:

$$\Phi(\boldsymbol{r}) = \Phi(\boldsymbol{r})\Big|_{\boldsymbol{r}=0} + \partial_i\Phi\Big|_{\boldsymbol{r}=0} x^i + \frac{1}{2!}\partial_i\partial_j\Phi\Big|_{\boldsymbol{r}=0} x^i x^j \tag{B.121}$$

But for keeping the distinction between test charge and external potential we need to make sure that Φ is not actually sourced by ρ itself: The Poisson-equation would stipulate that

$$\Delta\Phi = -4\pi\rho \tag{B.122}$$

and because the Laplace-operator acting on Φ is identical to the trace of the tensor of second derivatives of Φ, $\gamma^{ij}\partial_i\partial_j\Phi = \Delta\Phi$, it should not be contained in W. Therefore, one defines a traceless tensor

$$\partial_i\partial_j\Phi \quad\to\quad \partial_i\partial_j\Phi - \frac{\Delta\Phi}{3}\gamma_{ij} \tag{B.123}$$

by subtracting out the trace ΔΦ, such that the potential becomes

$$\Phi(\boldsymbol{r}) = \Phi(\boldsymbol{r})\Big|_{\boldsymbol{r}=0} + \partial_i\Phi\Big|_{\boldsymbol{r}=0} x^i + \frac{1}{6}\partial_i\partial_j\Phi\Big|_{\boldsymbol{r}=0}\left(3x^i x^j - r^2\gamma^{ij}\right). \tag{B.124}$$

Inclusion of the $r^2\gamma^{ij}$-term does not make any difference, because

$$x^i x^j\left(\partial_i\partial_j\Phi - \frac{\Delta\Phi}{3}\gamma_{ij}\right) = x^i x^j\partial_i\partial_j\Phi - \frac{\Delta\Phi}{3}\underbrace{\gamma_{ij}x^i x^j}_{=r^2} = x^i x^j\partial_i\partial_j\Phi \quad \text{as} \quad \Delta\Phi = 0. \tag{B.125}$$

The definitions of total charge q, dipole moment q^i and quadrupole moment q^{ij} are then identical to those discussed before, and the final expression of the interaction energy would be

$$W_{el} = \frac{1}{2}\int_V dV\, \rho(\boldsymbol{r})\Phi(\boldsymbol{r}) \simeq \frac{1}{2}q\Phi(\boldsymbol{r}) + \frac{1}{2}q^i\partial_i\Phi + \frac{1}{12}q^{ij}\partial_i\partial_j\Phi \tag{B.126}$$

with the interpretation that the interaction energy of nth order multipoles of the charge distribution is sensitive to the nth derivatives of Φ, and that they depend on the magnitude and relative orientation of the eigensystems of the tensors. This point of view is genuinely new, because the energy W can be changed by reorienting the charge distribution, in addition to displacing it. Additionally, the nth derivatives of the potential become measurable through their interaction energy with a multipole of order n, separate by order.

B.7 *Magnetic vector potential and gauging*

Magnetostatic problems, i.e. the computation of the magnetic fields for a given current density $\jmath^i$ with no contribution from time-varying electric fields require the solution of the fourth Maxwell-equation

$$\epsilon^{ijk}\partial_k \mathrm{H}_k = \frac{4\pi}{c}\jmath^i, \tag{B.127}$$

where this solution needs to fulfill the second Maxwell-equation $\partial_i \mathrm{B}^i = 0$ as a constraint. This constraint would be automatically fulfilled if B^i is derived from a magnetic potential A_i according to $\mathrm{B}^i = \epsilon^{ijk}\partial_j \mathrm{A}_k$, because $\partial_i \mathrm{B}^i = \epsilon^{ijk}\partial_i\partial_j \mathrm{A}_k = 0$, again through contraction of an antisymmetric with a symmetric object. Introducing the constitutive relation $\mathrm{H}_i = \mu_{ij}\mathrm{B}^j$ brings the fourth Maxwell-equation into the form

$$\epsilon^{ijk}\partial_j \mathrm{H}_k = \mu_{kl}\epsilon^{ijk}\partial_j \mathrm{B}^l = \mu_{kl}\epsilon^{ijk}\epsilon^{lmn}\partial_j\partial_m \mathrm{A}_n \tag{B.128}$$

which, for isotropic media with $\mu_{kl} = \gamma_{kl}/\mu$ leads to the Grassmann-relation

In an anisotropic medium, the Grassmann-algebra would read $\mu_{kl}\epsilon^{ijk}\epsilon^{lmn} = \mu^{im}\mu^{jn} - \mu^{in}\mu^{jm}$

$$= \frac{1}{\mu}\left(\gamma^{im}\gamma^{jn} - \gamma^{in}\gamma^{jm}\right)\partial_j\partial_m \mathrm{A}_n = \frac{1}{\mu}\left(\gamma^{im}\partial_m(\gamma^{jn}\partial_j \mathrm{A}_n) - \gamma^{in}(\gamma^{jm}\partial_j\partial_m \mathrm{A}_n)\right). \tag{B.129}$$

There exists the possibility to set the divergence $\gamma^{jn}\partial_j \mathrm{A}_n = 0$, called the Coulomb-gauge, showing that in fact a Poisson-type equation relates A_i and $\jmath^i$:

The Coulomb-gauge in a medium would be $\mu^{jn}\partial_j \mathrm{A}_n = 0$*...*

$$\Delta \mathrm{A}_i = -\frac{4\pi\mu}{c}\gamma_{ij}\jmath^j, \tag{B.130}$$

after multiplication of the equation with the inverse metric. Perhaps this is the right moment to emphasise that the "vector" potential A_i is in fact a linear form, and that the metric γ_{ij} is needed to convert the vector $\jmath^i$ to a linear form, to make the Poisson-equation notationally consistent.

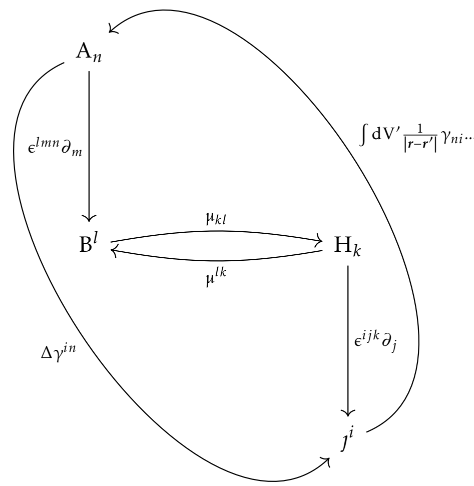
...and the field equation $\Delta \mathrm{A}_i = -4\pi\mu_{ij}\jmath^j/c$ *with* $\Delta = \mu^{ij}\partial_i\partial_j$

To illustrate the power of a gauge-assumption one could write eqn. (B.129) in matrix-vector notation, for the case of an isotropic medium and brushing slightly over the differences between vectors and linear forms, by using $\mathrm{A}^i = \gamma^{ij}\mathrm{A}_j$ as a vector,

$$\begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{pmatrix} \cdot \begin{pmatrix} \mathrm{A}_x \\ \mathrm{A}_y \\ \mathrm{A}_z \end{pmatrix} - \begin{pmatrix} \partial_x\partial_x & \partial_x\partial_y & \partial_x\partial_z \\ \partial_y\partial_x & \partial_y\partial_y & \partial_y\partial_z \\ \partial_z\partial_x & \partial_z\partial_y & \partial_z\partial_z \end{pmatrix} \cdot \begin{pmatrix} \mathrm{A}_x \\ \mathrm{A}_y \\ \mathrm{A}_z \end{pmatrix} = -\frac{4\pi\mu}{c}\begin{pmatrix} \jmath_x \\ \jmath_y \\ \jmath_z \end{pmatrix}. \tag{B.131}$$

While the first term, where A_i gets multiplied with a diagonal matrix that contains the Laplace-operator Δ, defines a one-to-one mapping of each component of A_i to its corresponding source $\jmath^i$, the association is broken by the second term, which is non-diagonal and supplies all kinds of mixed derivatives. But the assumption Coulomb-gauge makes these contributions vanish.

The assumption of Coulomb-gauge $\gamma^{jn}\partial_j \mathrm{A}_n = 0$ provides an astounding simplifaction, as each entry of A_i is sourced from each corresponding entry of j^i in three independent Poisson-equations. Without Coulomb-gauge, the term $\partial_i(\gamma^{jn}\partial_j A_n)$ would, as the gradient in the i-direction of the divergence of $\mathbf{A}$, couple all three equations. The interplay between the magnetic potential A_i (in Coulomb-gauge), the magnetic fields H_i, B^i and the source j^i is summarised by this diagram,



(B.132)

The physically measurable magnetic field B^i does not change under gauge transforms

$$A_i \rightarrow A_i + \partial_i \chi \tag{B.133}$$

because

$$B^i = \epsilon^{ijk}\partial_j A_k \rightarrow \epsilon^{ijk}\partial_j(A_k + \partial_k\chi) = \epsilon^{ijk}\partial_j A_k + \underbrace{\epsilon^{ijk}\partial_j\partial_k\chi}_{=0} = B^i \tag{B.134}$$

using that the gradient of a scalar field is always curl-free, $\epsilon^{ijk}\partial_j\partial_k$ vanishes as a contraction between an antisymmetric and symmetric tensor. This implies that the potential is only determined up to the gradient $\partial_i\chi$ of a scalar field χ (the gauge field). A particularly constructed field $\partial_i\chi$ can always be added onto A_i for computational convenience, without ever changing the actually measurable field B^i. This convenience might be the assumption of a gauge condition, for instance $\gamma^{ij}\partial_i A_j = 0$ (called Coulomb-gauge), which is necessary to have Poisson-type potential problems in magnetostatics.

The Coulomb-gauge condition transforms as

$$\gamma^{ij}\partial_i \mathrm{A}_j = 0 \rightarrow \gamma^{ij}\partial_i(\mathrm{A}_j + \partial_j \chi) = \gamma^{ij}\partial_i \mathrm{A}_j + \underbrace{\gamma^{ij}\partial_i\partial_j}_{=\Delta} \chi = 0. \tag{B.135}$$

If the vector potential A_i should be free of any divergence, one can construct χ as a solution to the Poisson-type equation

$$\Delta\chi = -\gamma^{ij}\partial_i \mathrm{A}_j, \tag{B.136}$$

effectively sourcing the gauge function χ with the yet nonzero divergence of the vector potential. It has, due to the Green-theorems, always a unique solution. Applying the gauge-transformation with this gauge field χ effectively cleans up the vector potential and makes it perfectly divergence-free. We can always assume that this has already been taken care of, just by writing $\gamma^{ij}\partial_i \mathrm{A}_j = 0$.

C.1 *Potentials and wave equations*

Working with static fields was a tremendous simplification of the Maxwell-equations and yielded, at least under the assumption of Coulomb-gauge, Poisson-type relations between the potentials Φ and A_i and the sources ρ and $\jmath^i$, with easily computable fields E_i and B^i. In taking the detour over the potentials one enables the full toolkit around Green-functions including the treatment of boundary conditions. But the existence of potentials, clearly at this point unrelated to energies as in the electrostatic case, follows from the homogeneous Maxwell-equations in a much more general argument: The second Maxwell-equation $\partial_i B^i = 0$ suggests that there is a vector field A_i with $B^i = \epsilon^{ijk}\partial_j A_k$, as then $\partial_i B^i = \epsilon^{ijk}\partial_i\partial_j A_k = 0$ is automatically fulfilled. Consequently, the induction law $\epsilon^{ijk}\partial_j E_k + \partial_{ct} B^i = 0$ becomes $\epsilon^{ijk}\partial_j E_k + \partial_{ct}\epsilon^{ijk}\partial_j A_k = \epsilon^{ijk}\partial_j(E_k + \partial_{ct} A_k) = 0$, suggesting a potential Φ with $E_i + \partial_{ct} A_i = -\partial_i\Phi$ (the minus-sign is conventional).

Therefore, the homogeneous Maxwell-equations ensure the existence of potentials in the general case, which again are only determined up to a gauge transform: As before, we write $A_i \to A_i + \partial_i\chi$ (which leaves B^i invariant) and investigate the necessary changes to Φ: The electric field E_i is gauge-invariant only if

$$E_i = -\partial_i\Phi - \partial_{ct}A_i \to -\partial_i\Phi + \underbrace{\partial_{ct}\partial_i\chi}_{\text{for consistency}} - \partial_{ct}(A_i + \partial_i\chi) = E_i \tag{C.137}$$

i.e. if we include an additional term $\partial_{ct}\chi$, implying the transformation rule

$$\Phi \to \Phi - \partial_{ct}\chi \quad \text{alongside} \quad A_i \to A_i + \partial_i\chi \tag{C.138}$$

for consistency, keeping in mind that partial derivatives interchange, $\partial_{ct}\partial_i\chi = \partial_i\partial_{ct}\chi$.

While the homogeneous Maxwell-equations safeguard the existence of potentials, the inhomogeneous Maxwell-equations couple the fields to the charges, be it static or dynamic. But while the homogeneous Maxwell-equations make statements about the observable fields E_i and B^i and derive them from potentials Φ and A_i, the coupling to sources is clarified by the inhomogeneous Maxwell-equations in terms of the auxiliary fields D^i and H_i. Hence, constitutive relations are needed.

In fact, the first Maxwell-equation makes a statement about the divergence of D^i, which is given in terms of the potentials by $E_i = -\partial_i\Phi - \partial_{ct}A_i$, followed by $D^i = \epsilon^{ij}E_j = \epsilon\gamma^{ij}E_j$, where we assume an isotropic medium. Consequently,

$$\partial_i D^i = \epsilon\gamma^{ij}\partial_i E_j = -\epsilon\gamma^{ij}\partial_i\partial_j\Phi - \epsilon\gamma^{ij}\partial_{ct}\partial_i A_j = 4\pi\rho \tag{C.139}$$

where we recover the conventional Poisson-equation $\epsilon\Delta\Phi = -\epsilon\gamma^{ij}\partial_i\partial_j\Phi = -4\pi\rho$ in Coulomb-gauge, $\gamma^{ij}\partial_i A_j = 0$. The fourth Maxwell-equation links the magnetic field H_i to $\jmath^i$ and the time derivative of the electric fields $\partial_{ct}D^i$, implying with $B^i = \epsilon^{ijk}\partial_j A_k$ and the constitutive relation $H_i = \mu_{ij}B^j = \gamma_{ij}B^j/\mu$

$$\epsilon^{ijk}\partial_j H_k = +\partial_{ct}D^i + \frac{4\pi}{c}\jmath^i \quad \to \quad \frac{1}{\mu}\epsilon^{ijk}\partial_j\gamma_{kl}\epsilon^{lmn}\partial_m A_n = +\partial_{ct}\epsilon\gamma^{ij}E_j + \frac{4\pi}{c}\jmath^i. \tag{C.140}$$

The contracted Levi-Civita symbol can be expanded in terms of the Grassmann-idenity,

$$\frac{1}{\mu}\gamma_{kl}\epsilon^{ijk}\epsilon^{lmn}\partial_j\partial_m A_n = \frac{1}{\mu}\left(\gamma^{im}\gamma^{jn} - \gamma^{in}\gamma^{jm}\right)\partial_j\partial_m A_n = \frac{1}{\mu}\left(\gamma^{im}\partial_m\left[\gamma^{jn}\partial_j A_n\right] - \gamma^{in}\left[\gamma^{jm}\partial_j\partial_m A_n\right]\right) \tag{C.141}$$

where one recognises the Coulomb-gauge term in the first and the Laplace-operator in the second square bracket. Substitution of the expression $E_i = -\partial_i\Phi - \partial_{ct}A_i$ on the right side leads to

$$\partial_{ct}\epsilon\gamma^{ij}E_j + \frac{4\pi}{c}\jmath^i = -\partial_{ct}\epsilon\gamma^{ij}\partial_j\Phi - \epsilon\partial_{ct}^2\gamma^{ij}A_j + \frac{4\pi}{c}\jmath^i \tag{C.142}$$

By assuming a different gauge condition, namely Lorenz-gauge[1]

$$\epsilon\partial_{ct}\Phi + \frac{1}{\mu}\gamma^{ij}\partial_i A_j = 0 \tag{C.143}$$

the two field equations decouple into a perfectly symmetric shape. Starting with eqn. (C.139), one obtains by substitution of the Lorenz-gauge condition

$$-\gamma^{ij}\partial_i\partial_j\Phi + \epsilon\mu\partial_{ct}^2\Phi = \frac{4\pi}{\epsilon}\rho, \tag{C.144}$$

i.e. a perfectly viable wave equation for Φ, sourced by ρ/ϵ. The same procedure applied to eqns. (C.141) and (C.142) leads likewise to a wave equation,

$$-\gamma^{jm}\partial_j\partial_m\gamma^{in}A_n + \epsilon\mu\partial_{ct}^2\gamma^{ij}A_j = \frac{4\pi}{c}\mu\jmath^i \tag{C.145}$$

With the definition of the d'Alembert-operator

$$\Box = \epsilon\mu\partial_{ct}^2 - \Delta \tag{C.146}$$

as a generalisation to the Laplace-operator Δ for dynamic situations, the two equations can be written as

$$\Box\Phi = \frac{4\pi}{\epsilon}\rho \quad\text{and}\quad \Box A_i = \frac{4\pi}{c}\mu\gamma_{ij}\jmath^j \tag{C.147}$$

and become two decoupled linear partial hyperbolic differential equations, providing 4 relations between 4 sources and 4 potentials, all decoupled by virtue of the Lorenz-gauge condition.

[1] The Lorenz-gauge is named after Ludvig Lorenz while the Lorentz-transformation was proposed by Hendrik Antoon Lorentz, hence the different spelling.

Differential equations involving the d'Alembert-operator typically have wave-like solutions, propagating at the velocity c, in this case modified to $c/\sqrt{\epsilon\mu}$, where one recognises $n = \sqrt{\epsilon\mu}$ as the index of refraction:

$$\epsilon\mu\partial_{ct}^2 = \epsilon\mu\frac{\partial^2}{\partial(ct)^2} = \frac{\epsilon\mu}{c^2}\frac{\partial^2}{\partial t^2} = \left(\frac{c}{\sqrt{\epsilon\mu}}\right)^{-2}\frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial(c't)^2} = \partial_{c't}^2 \quad \text{where} \quad c' = \frac{c}{\sqrt{\epsilon\mu}} \tag{C.148}$$

The gauge function χ for achieving Lorenz-gauge can be computed by considering the transformation of the expression $\epsilon\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j/\mu = 0$:

$$\epsilon\partial_{ct}\Phi + \frac{1}{\mu}\gamma^{ij}\partial_i A_j \rightarrow \epsilon\partial_{ct}\left(\Phi - \partial_{ct}\chi\right) + \frac{1}{\mu}\gamma^{ij}\partial_i\left(A_j + \partial_j\chi\right) =$$
$$\epsilon\partial_{ct}\Phi + \frac{1}{\mu}\gamma^{ij}\partial_i A_j - \epsilon\partial_{ct}^2\chi + \frac{1}{\mu}\Delta\chi = 0 \tag{C.149}$$

which is equivalent to

$$\Box\chi = \epsilon\mu\partial_{ct}^2\chi - \Delta\chi = \epsilon\mu\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j \tag{C.150}$$

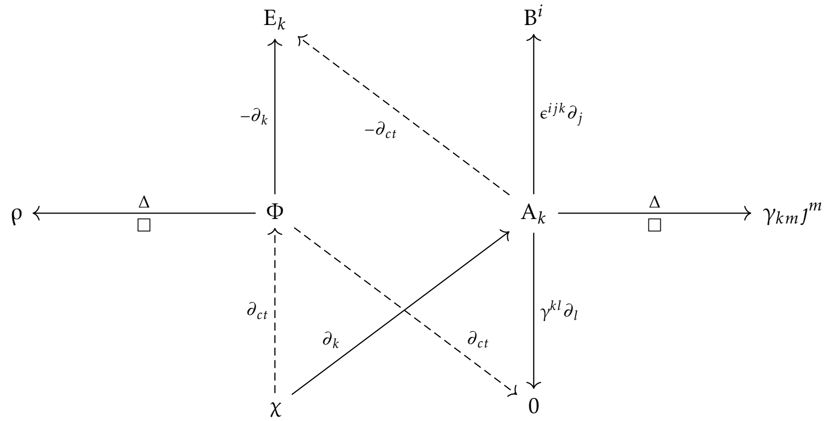
This is a wave-equation for χ, sourced by a possibly nonzero $\epsilon\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j/\mu$. As a hyperbolic partial linear differential equation, it has again a unique solution for χ, such that Lorenz-gauge can be imposed. Determining χ through $\Delta\chi = \gamma^{ij}\partial_i A_j/\mu$ for Coulomb-gauge in the static case and $\Box\chi = \epsilon\mu\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j$ for Lorenz-gauge in the dynamic case are completely analogous.

It is important to realise that the gauge freedom only provides a mathematical convenience for computing the potentials from the sources, and it can be used to set terms in the potential equations to zero. Nowhere there is anything physical happening: Purely by the act of imagining a new gauge condition the physically observable fields can not change. In addition, it is just practicality that persuades us to use Coulomb-gauge for the static case and Lorenz-gauge for the dynamical case, and not a physical requirement. In fact, it is perfectly reasonable to use the Coulomb-gauge $\gamma^{ij}\partial_i A_j = 0$ for the dynamical equations. Then, eqns. (C.139) and (C.141) become

$$\Delta\Phi = -\frac{4\pi}{\epsilon}\rho \quad \text{and} \quad \Delta A_i - \partial_{ct}^2 A_i = -\frac{4\pi}{c}\mu\gamma_{ij}j^j + \epsilon\partial_i\partial_{ct}\Phi \tag{C.151}$$

and deserve some explanation: The Poisson-equation provides an instantaneously changing Φ at any distance from the dynamically changing source ρ, while there is a wave-equation linking A_i to $\gamma_{ij}j^i$. But A_i depends as well on $\partial_i\partial_{ct}\Phi$ as a dynamic, vectorial source, hence the two equations are not yet fully decoupled. Coulomb-gauge might still be attractive though, because of the particularly easy expression for Φ!

The relationship between source, potential and fields are summarised for the case of static fields in Coulomb-gauge and for the dynamical case in Lorenz-gauge, where additional terms are indicated by dashed arrows:



(C.152)

The fields E_i and B^i are obtained from the potentials Φ and A_i by differentiation, and applying a second differentiation gives the sources ρ and $\jmath^i$. The direct path from the potentials to the sources is given by application of the Δ. The dynamic case is slightly more complicated, as E_i obtains a contribution $-\partial_{ct}A_i$ and as $\epsilon^{ijk}\partial_j H_k$ not only depends on $\jmath^i$, but also on $\partial_{ct}D^i$. The gauge function χ transforms only A_i in the static case, but both A_i and Φ in the dynamical case.

While we already know the Green-function inverting Δ from electrostatic and magnetostatic potentials and have encountered a systematic way of its construction, we now have to turn to $\Box$ and find a suitable time-dependent Green-function: the Liénard-Wiechert potentials.

C.2 *Solving the wave equation for potentials*

Intuitively it is clear that a changed charge distribution does not immediately affect the fields at any distance, but that there needs to be some time passing until the field configuration has adjusted itself to changes in the source distribution. For this purpose, let's assume Lorenz-gauge $\epsilon\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j/\mu = 0$ such that the field equations become

$$\Box\Phi = 4\pi\rho \quad \text{and} \quad \Box A_i = \frac{4\pi}{c}\gamma_{ij}\jmath^j \tag{C.153}$$

These equations are decoupled hyperbolic partial differential equations, with the charge density ρ and the current density $\jmath^i$ as sources. Clearly, in vacuum ρ and $\jmath^i$ vanish, such that one falls back onto two homogeneous PDEs

$$\Box\Phi = 0 \quad \text{as well as} \quad \Box A_i = 0 \tag{C.154}$$

which can be solved with a plane wave ansatz

$$\Phi, A_i \propto \exp\left(\pm i(\omega t - k_i r^i)\right). \tag{C.155}$$

Substitution yields for both Φ and A_i the result that

$$\Box \exp\left(\pm \mathrm{i}(\omega t - k_k r^i)\right) = \left[\left(\pm\frac{\mathrm{i}\omega}{c}\right)^2 - \gamma^{ab}\,(\mp \mathrm{i}k_a)\,(\mp \mathrm{i}k_b)\right]\exp\left(\pm \mathrm{i}(\omega t - k_i r^i)\right) = \left[-\left(\frac{\omega}{c}\right)^2 + \gamma^{ab}k_a k_b\right]\exp\left(\pm \mathrm{i}(\omega t - k_i r^i)\right) = 0 \quad \text{(C.156)}$$

i.e. the plane wave is a valid solution as long as the dispersion relation

$$\omega^2 = c^2\gamma^{ab}k_a k_b = (ck)^2 \quad \rightarrow \quad \omega = \pm ck \quad \text{(C.157)}$$

is fulfilled, which requires a strict proportionality between angular frequency ω and wave number k_a, with the speed of light c as the constant of proportionality. With this particular dispersion relation one can immediately show that the phase and group velocities are identical and have the value of c:

$$\upsilon_{\mathrm{ph}} = \frac{\omega}{k} = c = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \upsilon_{\mathrm{gr}} \quad \text{(C.158)}$$

which implies that wave packets in Φ and A_i propagate dispersion-free, i.e. without changing their shape. But perhaps more importantly, the wave equations suggest that excitations of the fields travel at a finite speed c in the potentials Φ and A_i (at least in Lorenz-gauge, the statement would not be true in Coulomb-gauge!).

C.3 *Wave equation for fields*

While propagation and the form of the propagation equations depends on the level of the potentials A_i and Φ on the assumed gauge, the fields E_i and B^i as physical observables can never depend on a certain gauge and always exhibit propagation at the speed of light c. In a vacuum situation with $\jmath^i = 0$ as well as $\rho = 0$ both fields are divergence-free $\partial_i D^i = \partial_i B^i = 0$ and the rotations are defined, up to a sign arising from duality invariance, by

$$\epsilon^{ijk}\partial_j E_k = -\partial_{ct}B^i \quad \text{and} \quad \epsilon^{ijk}\partial_j H_k = +\partial_{ct}D^i. \quad \text{(C.159)}$$

Taking a further rotation of any of the two equations, using rot rot $= \nabla$div $- \Delta$, setting the divergence-term to zero and substituting the time derivative of the other equation leads to

$$\left(\partial_{ct}^2 - \Delta\right)E_i = \Box E_i = 0 \quad \text{and, in parallel,} \quad \left(\partial_{ct}^2 - \Delta\right)H_i = \Box H_i = 0 \quad \text{(C.160)}$$

Including a source term for the wave equations for the fields themselves is a bit complicated, but we'll return to that issue later.

i.e. perfectly symmetric wave equations for the electric and magnetic fields, with excitations travelling at the speed of light c. The symmetry in the shape of the equations is perhaps not too surprising, as one can always replace the fields in a duality transform $E_i \rightarrow H_i$ and $H_i \rightarrow -E_i$ valid in vacuum. Solving the wave equations with a plane wave ansatz $\propto \exp\left(\pm \mathrm{i}(\omega t - k_i r^i)\right)$ is perfectly general: Due to the linearity of the PDEs, any field configuration can be written as a superposition of plane waves that solve the wave equation.

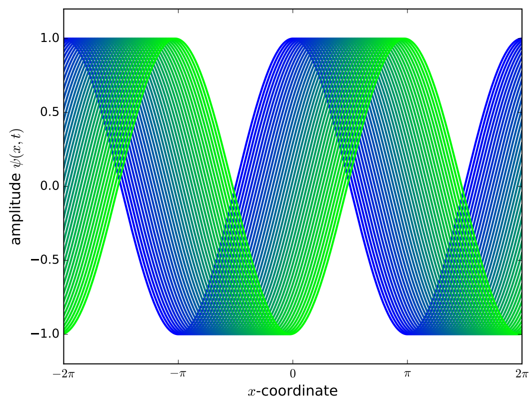


Figure 6: Harmonic wave as a function of x; the shading indicates evolution with time t.

Substituting the plane wave-ansatz into the divergences shows that

$$\gamma^{ij}k_i\mathrm{E}_j = 0 \quad \text{and} \quad \gamma^{ij}k_i\mathrm{H}_j = 0 \tag{C.161}$$

implying that the fields are transverse, i.e. the amplitudes are perpendicular to the direction of propagation, and substituting into the rotation equations suggests

$$\epsilon^{ijk}k_j\mathrm{E}_k = -\frac{\omega}{c}\mathrm{B}^i = -\frac{\omega}{\mu c}\gamma^{ij}\mathrm{H}_k \quad \text{as well as} \quad \epsilon^{ijk}k_j\mathrm{H}_k = +\frac{\omega}{c}\mathrm{D}^i = +\frac{\omega\epsilon}{c}\gamma^{ij}\mathrm{E}_j \tag{C.162}$$

such that the amplitudes of the fields themselves are perpendicular to each other. Please note that the statements of transversality and perpendicularity can not be independent: Pictorially, there is simply no other direction in which k could point: Multiplying the latter two equations with k_i already implies that $\gamma^{ij}k_i\mathrm{H}_j = \gamma^{ij}k_i\mathrm{E}_k = 0$. It is quite instructive to multiply with the linear forms H_i and E_i, leading to

$$\epsilon^{ijk}\mathrm{H}_ik_j\mathrm{E}_k = -\frac{\omega}{c}\mathrm{H}_i\mathrm{B}^i \quad \text{as well as} \quad \epsilon^{ijk}\mathrm{E}_ik_j\mathrm{H}_k = +\frac{\omega}{c}\mathrm{E}_i\mathrm{D}^i \tag{C.163}$$

showing that the volume of the rectangular cuboid spanned by the linear forms E_i, H_i and k_i is proportional to the energy densities, which are equal for a plane wave.

While the amplitudes E_i and H_i are always perpendicular to the direction of propagation, the analogous statement for vector potential A_i is only true under Coulomb-gauge, $\gamma^{ij}k_i\mathrm{A}_j = 0$: This is the reason why sometimes one refers to this gauge condition as transverse gauge. It is quite funny to go through all vector orientations for a duality transform. As plane electromagnetic waves are vacuum solutions, this transform must yield a physically sensible field configuration: Even the fact that k_i, E_i and H_i from a right-handed system in the sense that $\epsilon^{ijk}k_i\mathrm{E}_j\mathrm{H}_k$ is positive is conserved under duality transforms.

Fig. 6 shows how a wave of the type $\exp(\pm\mathrm{i}(\omega t - k_ir^i))$ propagates: Not only is it an oscillation in t at fixed r^i and an oscilation in r^i at fixed t, but the two are coupled: Defining the phase velocity $\upsilon_{\mathrm{ph}} = \omega/k$ makes the argument assume the form $\upsilon_{\mathrm{ph}} t - r$, and moving along with this velocity with the wave an observer would always perceive the phase function $\phi = \omega t - k_i r^i$ to be constant. The phase function ϕ has an interesting geometric shape as $k_i r^i - \omega t$ = const corresponds to the Hesse normal form of a plane, specifically in our case the plane of constant phase. As time progresses, this plane of constant phase moves along its surface normal k_i, which allows the identification of the wave "vector" k_i (actually a linear form) as the direction of propagation.

The wave "vector" should better be a linear form, as in quantum mechanics it is related to the momentum p_i, likewise a linear form, by $p_i = \hbar k_i$!

C.4 *Electromagnetic waves in matter and the telegraph equation*

Electromagnetic waves in matter experience two effects: Firstly, ϵ and μ can differ from one, such that one has to work with $D^i = \epsilon^{ij} E_j$ and $B^i = \mu^{ij} H_j$ in a potentially anisotropic way, and secondly, the electric field E_i might be able to move the charge carries in the medium, giving rise to a current density $\jmath^i$, where the two are related by Ohm's law. It reads in its differential formulation

$$\jmath^i = \sigma^{ij} E_j \tag{C.164}$$

with the conductivity tensor σ^{ij}. As in the case of the dielectric constant ϵ and the permeability μ, the conductivity σ is scalar only in the case of isotropic media (perhaps one can imagine a somehow layered material as a counter example, in which the charges are movable at different rates in the different directions), and a linear relationship is essentially a first order approximation.

Faraday's induction law in an isotropic medium assumes the form

$$\epsilon^{ijk} \partial_j E_k = -\partial_{ct} B^i = -\mu \gamma^{ij} \partial_{ct} H_j \tag{C.165}$$

and Ampère's law takes on the shape

$$\epsilon^{ijk} \partial_j H_k = +\partial_{ct} D^i + \frac{4\pi}{c} \jmath^i = +\epsilon \gamma^{ij} \partial_{ct} E_j + \frac{4\pi\sigma}{c} \gamma^{ij} E_j \tag{C.166}$$

so that taking the time derivative of the first equation, and the rotation of the second equation, again by using the Grassmann-relation leads to a wave equation with a damping term, the so-called telegraph equation

$$\left(\epsilon\mu\partial_{ct}^2 - \Delta\right) H_i = \Box H_i = -\frac{4\pi\sigma\mu}{c} \partial_{ct} H_i \tag{C.167}$$

The effective speed of propagation c' is given by

$$c' = \frac{c}{\sqrt{\epsilon\mu}} \simeq \frac{c}{\sqrt{\epsilon}} \tag{C.168}$$

as effectively all known transparent media have permeabilities close to one. The latter relation suggests that the refractive index n is given by $\sqrt{\epsilon}$, relating electrical to optical properties of a medium. The damping is determined by the conductivity σ: Non-conductive media do not show any attenuation of incident electromagnetic waves, but if the conductivity is nonzero, the motion of the charges in the medium dissipate the energy of the electromagnetic waves.

Dimensional analysis shows that the term

$$\frac{4\pi\sigma\mu}{c} = L_{\mathrm{att}} \tag{C.169}$$

must have units of a length scale L_{att} on which the amplitude of the wave decreases by a factor $\exp(-1)$.

C.5 *Energy transport and the Poynting vector*

We have already seen that the electric and magnetic fields are real in the sense that they accelerate test charges and contain energy at the densities

$$w_{\mathrm{el}} = \frac{E_i D^i}{8\pi} \quad \text{and analogously,} \quad w_{\mathrm{mag}} = \frac{H_i B^i}{8\pi} \tag{C.170}$$

The corresponding energies, obtained by integration over space, would from a combined energy conservation law together with the mechanical energies. As the fields can dynamically evolve, the questions how energy is conserved by a dynamically evolving field configuration and how it is transported through space arises naturally.

A good starting point are the two inhomogeneous Maxwell-equations that contain time derivatives:

$$\epsilon^{ijk}\partial_j E_k = -\partial_{ct} B^i \quad \text{as well as} \quad \epsilon^{ijk}\partial_j H_k = +\partial_{ct} D^i + \frac{4\pi}{c} \jmath^i. \tag{C.171}$$

Multiplying the first equation with H_i and the second equation with E_i in the sense of a scalar product and subsequent subtraction yields

$$E_i\epsilon^{ijk}\partial_j H_k - H_i\epsilon^{ijk}\partial_j E_k = \frac{4\pi}{c} E_i \jmath^i + \underbrace{E_i\partial_{ct} D^i}_{=\frac{1}{2}\partial_{ct}(E_i D^i)} + \underbrace{H_i\partial_{ct} B^i}_{=\frac{1}{2}\partial_{ct}(H_i B^i)} \tag{C.172}$$

where the reshaping in the last two terms with the constitutive relation suggests substitution of the energy densities:

$$4\pi\partial_{ct} w_{\mathrm{el}} = \frac{1}{2}\partial_{ct}(E_i D^i) = \frac{1}{2}\left(\partial_{ct} E_i \cdot D^i + E_i \partial_{ct} D^i\right) =$$
$$\frac{\epsilon^{ij}}{2}\left(\partial_{ct} E_i \cdot E_j + E_i\partial_{ct} E_j\right) = E_i \partial_{ct}\epsilon^{ij} E_j = E_i\partial_{ct} D^i, \tag{C.173}$$

relying on the symmetry of the dielectric tensor ϵ^{ij}, and likewise

$$4\pi\partial_{ct} w_{\mathrm{mag}} = \frac{1}{2}\partial_{ct}(H_i B^i) = \frac{1}{2}\left(\partial_{ct} H_i \cdot B^i + H_i \partial_{ct} B^i\right) =$$
$$\frac{\mu^{ij}}{2}\left(\partial_{ct} H_i \cdot H_j + H_i\partial_{ct} H_j\right) = H_i \partial_{ct}\mu^{ij} H_j = H_i\partial_{ct} B^i, \tag{C.174}$$

for the magnetic fields with a symmetric permeability μ^{ij}. The left hand side of the equation can be written as

$$E_i\epsilon^{ijk}\partial_j H_k - H_i\epsilon^{ijk}\partial_j E_k = E_i\epsilon^{ijk}\partial_j H_k - H_k\epsilon^{kji}\partial_j E_i =$$
$$\epsilon^{ijk}(E_i\partial_j H_k + H_k\partial_j E_i) = \partial_j\epsilon^{ijk}E_i H_k = -\partial_i\epsilon^{ijk}E_j H_k \quad \text{(C.175)}$$

with renaming the indices $i \leftrightarrow k$ in the second term, before reordering $\epsilon^{kji} = \epsilon^{ikj} = -\epsilon^{ijk}$, with a cycling permutation in the first and an interchange in the second step. Defining the Poynting-vector P^i

Energy transport depends on the two linear forms E_i and H_i.

$$P^i = \frac{c}{4\pi}\epsilon^{ijk}E_j H_k \quad \text{(C.176)}$$

one arrives at the final result

$$\partial_i P^i = -E_i \jmath^i - \partial_t\left(w_{el} + w_{mag}\right) \quad \text{(C.177)}$$

The Gauß-theorem allows to recast this differential conservation law into integral form,

$$\int_V dV\,\partial_i P^i = \int_{\partial V} dS_i P^i = -\int_V dV\,E_i \jmath^i - \frac{d}{dt}\int_V dV\,\left(w_{el} + w_{mag}\right) \quad \text{(C.178)}$$

such that the change in energy content of the inside the volume V is given by two terms. The first term describes the energy flux integrated over the surface ∂V: If the Poynting-vector P^i has a nonzero divergence and points outwards, the energy content will decrease. The second term is attributed to the dissipation inside the volume: Introducing Ohm's law in differential form,

$$\jmath^i = \sigma^{ij}E_j \quad \text{(C.179)}$$

with the conductivity tensor σ^{ij}, the integral over the volume $V = A\ell$

$$\int_V dV\,E_i\jmath^i = \int_V dV\,\sigma^{ij}E_iE_j = \sigma\int_V dV\,\gamma^{ij}E_iE_j \simeq \sigma E^2 V = \underbrace{\frac{\sigma A}{\ell}}_{=1/R}(\underbrace{E\ell}_{=U})^2 = \frac{U^2}{R} \quad \text{(C.180)}$$

This is exactly the energy per time interval which is dissipated into heat inside the volume V. Alternatively, one could have replaced E_i instead of $\jmath^i$, leading to

$$\int_V dV\,E_i\jmath^i = \int_V dV\,\sigma_{ij}\jmath^i\jmath^j = \frac{1}{\sigma}\int_V dV\,\gamma_{ij}\jmath^i\jmath^j \simeq \frac{\jmath^2 V}{\sigma} = \underbrace{\frac{\ell}{A\sigma}}_{=R}(\underbrace{\jmath A}_{=I})^2 = RI^2 \quad \text{(C.181)}$$

Only if the conductivity σ vanishes, or the resistance R is infinite, the terms is inactive and the ideal energy conservation law is given by

$$\partial_i P^i = -\partial_t\left(w_{el} + w_{mag}\right) \quad \rightarrow \quad \int_{\partial V} dS_i\,P^i = -\frac{d}{dt}\int_V dV\,\left(w_{el} + w_{mag}\right) \quad \text{(C.182)}$$

C.6 *Momentum transport and the Poynting linear form*

Similarly to the Poynting-law for energy conservation of the electromagnetic field there is an associated momentum conservation law: Starting at the Lorentz-equation for a volume element that contains a charge density ρ and a current density $\jmath^i$ allows to express the rate of change of the volume's momentum

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \int_V \mathrm{dV} \left(\rho \mathrm{E}_i + \frac{1}{c}\epsilon_{ijk}\jmath^j \mathrm{B}^k\right) \tag{C.183}$$

which would fall back onto eqn. (A.1) by setting $\mathrm{d}q = \rho\mathrm{dV}$, $q = \int \mathrm{d}q = \int \mathrm{dV}\,\rho$.

As in the calculation for the energy density of the electromagnetic fields we can replace the charge density ρ and the current density $\jmath^i$ by using the two inhomogeneous Maxwell-equations $\partial_i \mathrm{D}^i = 4\pi\rho$ and $\epsilon^{jmn}\partial_m \mathrm{H}_n = \partial_{ct}\mathrm{D}^j + 4\pi/c\, \jmath^j$ leading to the change of the momentum associated with the fields themselves

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \frac{1}{4\pi}\int_V \mathrm{dV} \left(\mathrm{E}_i\, \partial_j \mathrm{D}^j + \epsilon_{ijk}\epsilon^{jmn}\partial_m \mathrm{H}_n \cdot \mathrm{B}^k - \partial_{ct}\mathrm{D}^j \cdot \mathrm{B}^k\right) \tag{C.184}$$

Aiming at making the expression more symmetric, it is clearly possible to add the term $\mathrm{H}_i \partial_j \mathrm{B}^j$ as $\partial_i \mathrm{B}^i = 0$, and replacing the last term $\partial_{ct}\mathrm{D}^j \cdot \mathrm{B}^k$ using the Leibnitz-rule according to

$$\partial_{ct}\left(\mathrm{D}^j \mathrm{B}^k\right) = \partial_{ct}\mathrm{D}^j \cdot \mathrm{B}^k + \mathrm{D}^j \partial_{ct}\mathrm{B}^k. \tag{C.185}$$

Then, the penultimate term requires the time-derivative of the magnetic field, which suggests to substitute the induction equation $\partial_{ct}\mathrm{B}^k = -\epsilon^{kmn}\partial_m \mathrm{E}_n$:

$$\epsilon_{ijk}\partial_{ct}\mathrm{D}^j \cdot \mathrm{B}^k = \partial_{ct}\left(\epsilon_{ijk}\mathrm{D}^j \mathrm{B}^k\right) - \epsilon_{ijk}\mathrm{D}^j \partial_{ct}\mathrm{B}^k = \partial_{ct}\left(\epsilon_{ijk}\mathrm{D}^j \mathrm{B}^k\right) + \epsilon_{ijk}\mathrm{D}^j \epsilon^{kmn}\partial_k \partial_m \mathrm{E}_n \tag{C.186}$$

⚠ *Momentum transport depends on the vectorial fields* D^i *and* B^i*.*

The formula suggests a Poynting linear form Y_i

$$\mathrm{Y}_i = \frac{c}{4\pi}\epsilon_{ijk}\mathrm{D}^j \mathrm{B}^k \tag{C.187}$$

analogous to the vector $\mathrm{P}^i = c/(4\pi)\, \epsilon^{ijk}\mathrm{E}_j \mathrm{H}_k$, but composed of the two vectorial fields D^i and B^i. The missing c suggests that it has units of a momentum density, and hence it describes the momentum content associated with the fields inside volume.

The expression for the momentum change presents itself in a wonderfully symmetric form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_i + \int_V \mathrm{dV}\, \mathrm{Y}_i\right) = \frac{1}{4\pi}\int_V \mathrm{dV} \left(\mathrm{E}_i \partial_j \mathrm{D}^j + \mathrm{H}_i \partial_j \mathrm{B}^j - \epsilon_{ijk}\epsilon^{kmn}\partial_m \mathrm{E}_n \cdot \mathrm{D}^j + \epsilon_{ijk}\epsilon^{jmn}\partial_m \mathrm{H}_n \cdot \mathrm{B}^k\right) \tag{C.188}$$

If the right side of this equation could be written as a divergence, one would recover the archetypical form of a continuity equation, this time for the momentum of the field configuration inside the volume V. As the left side of the equation is vectorial and because taking a divergence reduces the rank of a tensor by one, we are looking for a tensor of rank two on the right side. Treating the electric fields first, shows that

$$\begin{aligned} \mathrm{E}_i \partial_j \mathrm{D}^j - \epsilon_{ijk}\epsilon^{kmn}\partial_m \mathrm{E}_n \cdot \mathrm{D}^j &= \mathrm{E}_i \partial_j \mathrm{D}^j - \left(\delta_i^m \delta_j^n - \delta_j^m \delta_i^n\right)\partial_m \mathrm{E}_n \cdot \mathrm{D}^j = \\ \mathrm{E}_i \partial_j \mathrm{D}^j - \partial_i \mathrm{E}_j \cdot \mathrm{D}^j + \partial_j \mathrm{E}_i \cdot \mathrm{D}^j &= \underbrace{\mathrm{E}_i \partial_j \mathrm{D}^j + \partial_j \mathrm{E}_i \cdot \mathrm{D}^j}_{=\partial_j\left(\mathrm{E}_i \mathrm{D}^j\right)} - \underbrace{\partial_i \mathrm{E}_j \cdot \mathrm{D}^j}_{=\delta_j^i \partial_j\left(\mathrm{E}_k \mathrm{D}^k\right)/2}\ , \end{aligned} \tag{C.189}$$

after a reordering of terms. While the first combination is just an application of the Leibnitz-rule, the rewriting of the last term deserves a more thorough argument:

$$\begin{aligned} \partial_i \mathrm{E}_j \cdot \mathrm{D}^j = \delta_i^j \partial_j \mathrm{E}_k \cdot \mathrm{D}^k = \delta_i^j \epsilon_{km} \partial_j \mathrm{E}_k \cdot \mathrm{E}_m = \frac{1}{2}\delta_i^j \epsilon^{km} \partial_j \left(\mathrm{E}_k \mathrm{E}_m\right) = \\ \frac{1}{2}\delta_i^j \partial_j \left(\epsilon^{km} \mathrm{E}_k \mathrm{E}_m\right) = \frac{1}{2}\delta_i^j \partial_j \left(\mathrm{E}_k \mathrm{D}^k\right). \end{aligned} \tag{C.190}$$

The terms involving magnetic fields are treated in complete analogy up to a difference in sign, caused by the different contraction. This is quickly remedied by interchanging the indices $\epsilon^{ijk} = -\epsilon^{ikj}$:

$$\mathrm{H}_i \partial_j \mathrm{B}^j + \epsilon_{ijk}\epsilon^{jmn}\partial_m \mathrm{H}_n \cdot \mathrm{B}^k = \mathrm{H}_i \partial_j \mathrm{B}^j - \epsilon_{ikj}\epsilon^{jmn}\partial_m \mathrm{H}_n \cdot \mathrm{B}^k \tag{C.191}$$

The subsequent steps are identical:

$$\begin{aligned} \mathrm{H}_i \partial_j \mathrm{B}^j - \epsilon_{ikj}\epsilon^{jmn}\partial_m \mathrm{H}_n \cdot \mathrm{B}^k &= \mathrm{H}_i \partial_j \mathrm{B}^j - \left(\delta_i^m \delta_k^n - \delta_k^m \delta_i^n\right)\partial_m \mathrm{H}_n \cdot \mathrm{B}^k = \\ \mathrm{H}_i \partial_j \mathrm{B}^j - \partial_i \mathrm{H}_k \cdot \mathrm{B}^k + \partial_k \mathrm{H}_i \cdot \mathrm{B}^k &= \underbrace{\mathrm{H}_i \partial_j \mathrm{B}^j + \partial_k \mathrm{H}_i \cdot \mathrm{B}^k}_{=\partial_j\left(\mathrm{H}_i \mathrm{B}^j\right)} - \underbrace{\partial_i \mathrm{H}_k \cdot \mathrm{B}^k}_{=\delta_j^i \partial_j\left(\mathrm{H}_k \mathrm{B}^k\right)/2}\ . \end{aligned} \tag{C.192}$$

where an identical argument applies to the last term:

$$\begin{aligned} \partial_i \mathrm{H}_j \cdot \mathrm{B}^j = \delta_i^j \partial_j \mathrm{H}_k \cdot \mathrm{B}^k = \delta_i^j \mu_{km} \partial_j \mathrm{H}_k \cdot \mathrm{H}_m = \frac{1}{2}\delta_i^j \mu^{km} \partial_j \left(\mathrm{H}_k \mathrm{H}_m\right) = \\ \frac{1}{2}\delta_i^j \partial_j \left(\mu^{km} \mathrm{H}_k \mathrm{H}_m\right) = \frac{1}{2}\delta_i^j \partial_j \left(\mathrm{H}_k \mathrm{B}^k\right). \end{aligned} \tag{C.193}$$

Collecting all terms finally gives the sought-after divergence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_i + \int_{\mathrm{V}} \mathrm{dV}\ \mathrm{Y}_i\right) = \int_{\mathrm{V}} \mathrm{dV}\ \partial_j \mathrm{T}_i{}^j = \int_{\partial \mathrm{V}} \mathrm{dS}_j\ \mathrm{T}_i{}^j \tag{C.194}$$

where the Gauß-theorem was applied in the last step, yielding a surface integral over the Maxwell stress tensor $\mathrm{T}_i{}^j$

$$\mathrm{T}_i{}^j = \frac{1}{4\pi}\left(\mathrm{E}_i \mathrm{D}^j + \mathrm{H}_i \mathrm{B}^j - \frac{1}{2}\delta_i^j \left(\mathrm{E}_k \mathrm{D}^k + \mathrm{H}_k \mathrm{B}^k\right)\right) \tag{C.195}$$

The Maxwell-tensor is symmetric, $T_i^{\ j} = T_j^{\ i}$ in the case of isotropic media, but in general not: Examining $E_i D^j$, for instance, shows with the substitution

$$E_i D^i = \epsilon^{ij} E_i E_j = \epsilon_{ij} D^i D^j \tag{C.196}$$

that it can only be symmetric if ϵ^{ij} and $E_i E_j$, despite being both symmetric on their own, have coinciding eigensystems. This would be the case for isotropic media, as γ^{ij} and $E_i E_j$ are simultaneously diagonalisable. A straightforwardly mathematical condition would be a vanishing commutator $\left[\epsilon^{ij}, E_i E_j\right] = 0$.

It is striking that in an anisotropic medium the direction of energy transport and momentum transport are not collinear, as

$$Y_i = \frac{c}{4\pi}\epsilon_{ijk} D^j B^k = \frac{c}{4\pi}\epsilon_{ijk}\epsilon^{jm} E_m \mu^{kn} H_n \tag{C.197}$$

Forming the scalar product between the Poynting vector P^i and its associated linear form Y_i gives

$$Y_i P^i = \frac{c^2}{(4\pi)^2}\epsilon_{ijk}\epsilon^{imn} D^j B^k E_m H_n = \frac{c^2}{(4\pi)^2}\left(\delta^m_j \delta^n_k - \delta^m_k \delta^n_j\right) D^j B^k E_m H_n =$$
$$\frac{c^2}{(4\pi)^2}\left(D^m E_m B^n H_n - D^j H_j B^k E_k\right) = \frac{c^2}{(4\pi)^2}\left(\epsilon^{im} E_i E_m \mu^{jn} H_j H_n - \epsilon^{im} E_i H_m \mu^{jn} H_j E_n\right), \tag{C.198}$$

with the squared norms of the two fields in the first and the scalar products in the second term: This suggests that the scalar product $Y_i P^i$ is positive definite, as a result of the Cauchy-Schwarz inequality. After rewriting the expression in terms of the two constitutive tensors instead of the fields one arrives at

$$Y_i P^i = \frac{c^2}{(4\pi)^2}\epsilon^{im}\mu^{jn}\left(E_i H_j E_m H_n - E_i H_j E_n H_m\right) = \frac{c^2}{(4\pi)^2}\left(\epsilon^{im}\mu^{jn} - \epsilon^{in}\mu^{jm}\right) E_i H_j E_m H_n \tag{C.199}$$

For a plane wave with perpendicular electric and magnetic fields one would obtain, under the assumption of an isotropic medium, a vanishing second term, yielding the largest possible result for $Y_i P^i$, which indicates a parallel momentum and energy transport.

The trace $\mathrm{tr}(T) = \delta^i_j T_i^{\ j} = T_i^{\ i}$ computes to the negative energy density of the fields, as

$$T_i^{\ i} = \frac{1}{4\pi}\left(E_i D^i + H_i B^i - \frac{\delta^j_i \delta^i_j}{2}\left(E_k D^k + H_k B^k\right)\right) = -\frac{1}{8\pi}\left(E_i D^i + H_i B^i\right) = -(w_{\mathrm{el}} + w_{\mathrm{mag}}) \tag{C.200}$$

as $\delta^j_i \delta^i_j = \delta^i_i = 3$. To be honest, this result can only be understood later, when we derive the Maxwell-tensor for electrodynamics as a relativistic field theory.

Looking at the mechanical aspect of the continuity equation for the momentum density as the change of momentum needs to be equal to the force acting on the volume element, and because dp_i is given as

$$\mathrm{d}\dot{p}_i = \mathrm{T}_i{}^j \mathrm{dS}_j \quad \rightarrow \quad \mathrm{T}_i{}^j = \frac{\partial \dot{p}_i}{\partial \mathrm{S}_j}, \tag{C.201}$$

one would associate $\mathrm{T}_i{}^j$ with a force per unit area: Those elements are referred to as stresses, of which the isotropic component is called radiation pressure. Depending on the field configuration, the stresses into different coordinate directions do not need to be equal. Commonly, one would expect radiation pressure to be exerted in the direction of propagation of an electromagnetic wave, but not perpendicularly to it. On the other hand, an isotropic superposition of plane electromagnetic waves, as for instance in a blackbody, can be assigned a radiation pressure. The combined term on the left side of the equation is the mechanical momentum p_i of the matter inside the volume V and the volume-integrated Poynting linear form Y_i as the momentum content of the electromagnetic field.

We will see that the four entities energy density $w = w_{\mathrm{el}} + w_{\mathrm{mag}}$, Poynting vector P^i or energy flux density, Poynting linear form Y_i or momentum density and the Maxwell stress tensor $\mathrm{T}_i{}^j$ can be assembled into a larger object, the energy momentum-tensor $\mathrm{T}_\mu{}^\nu$:

$$\mathrm{T}_\mu{}^\nu = \left(\begin{array}{c|c} w & \mathrm{Y}_i \\ \hline \mathrm{P}^j & \mathrm{T}_i{}^j \end{array}\right), \tag{C.202}$$

which will, when a combined derivative $\partial_\nu = (\partial_{ct}, \partial_j)$ is applied to it, yield energy conservation in the first column, and the three components of momentum conservation in the second, third and fourth columns. All conservation laws would then follow jointly from the divergence $\partial_\nu \mathrm{T}_\mu{}^\nu = 0$, for media of zero conductivity, and the entire tensor is traceless, $\delta^\mu_\nu \mathrm{T}_\mu{}^\nu = \mathrm{T}_\mu{}^\mu = 0 = w + \delta^i_j \mathrm{T}_i{}^j = w + \mathrm{T}_i{}^i$.

In summary, there is a clear notion of energy and momentum conservation in the electromagnetic field. One can associate energy and momentum densities to any field configuration, and as the configuration evolves dynamically, energy and momentum is transported through space in a way that is described by continuity equations. Possible dissipation can be described by Ohm's law, and would convert field energy into heat. The Poynting-vector plays the role as energy flux and is constructed from the linear forms E_i and H_i, while the transport of momentum density is encapsulated in the related linear form Y_i, which depends on the two vectors D^i and B^i. It is straightforward to see and not unexpected that for a plane electromagnetic wave the energy transport proceed along the wave vector, as P^i is collinear with k_i, which in turn is poynting (pardon me!) into the direction $\epsilon^{ijk}\mathrm{E}_j\mathrm{H}_k$. In metric spaces or spacetimes it is always possible to write the Maxwell stress-tensor and the energy-momentum tensor with one type of index, covariant for instance,

$$\mathrm{T}_{ij} = \gamma_{jk}\mathrm{T}_i{}^k \quad \text{and} \quad \mathrm{T}_{\mu\nu} = \eta_{\nu\alpha}\mathrm{T}_\mu{}^\alpha, \tag{C.203}$$

such that the traces read $\gamma^{ij}\mathrm{T}_{ij}$ and $\eta^{\mu\nu}\mathrm{T}_{\mu\nu}$, and the divergences $\gamma^{ai}\partial_a \mathrm{T}_{ij} = 0$ and $\eta^{\alpha\mu}\partial_\alpha \mathrm{T}_{\mu\nu} = 0$.

C.7 *Time-dependent Green-functions and retardation*

Clearly, the propagation speed of excitations in the electromagnetic field is finite, so any change in the source configuration is not perceived instantaneously at any point at nonzero distance: In fact, the changed field configuration only arrives after a time

$c\Delta t = \Delta r$ at distance Δr, which is referred to as retardation. The same is true for the potentials Φ and A_i if one assumes Lorenz-gauge $\epsilon\partial_{ct}\Phi + \gamma^{ij}\partial_i A_j/\mu = 0$, because then the wave equations for the potentials

$$\Box\Phi = 4\pi\rho \quad \text{and} \quad \Box A_i = \frac{4\pi}{c}\gamma_{ij}\jmath^j \tag{C.204}$$

are identical to those of the fields E_i and B^i. This particular form of an inhomogeneous wave equation, where we always verified that the homogeneous differential equation is solved by a plane wave, is referred to as the Helmholtz differential equation

$$\Box\psi(\boldsymbol{r}, t) = \left(\partial_{ct}^2 - \Delta\right)\psi(\boldsymbol{r}, t) = 4\pi q(\boldsymbol{r}, t) \tag{C.205}$$

where ψ could be either of the potentials Φ and A_i, and q the corresponding source, i.e. ρ or $\gamma_{ij}\jmath^j$. The Helmholtz differential equation is a hyperbolic linear partial differential equation of second order with an inhomogeneity. As a linear differential equation, a suitable solution strategy would be a Green-function, that depends both on space and time coordinates:

$$\Box G(\boldsymbol{r} - \boldsymbol{r}', t - t') = 4\pi\delta_D(\boldsymbol{r} - \boldsymbol{r}')\delta_D(t - t') \tag{C.206}$$

As before, the Green-function is the formal solution for the potential at $\boldsymbol{r}$ and t to a point-like source existing at $\boldsymbol{r}'$ and t', such that

$$\Box\psi(\boldsymbol{r}, t) = 4\pi q(\boldsymbol{r}, t) \quad \rightarrow \quad \psi(\boldsymbol{r}, t) = \int dV' \int dt'\, G(\boldsymbol{r} - \boldsymbol{r}', t, -t')q(\boldsymbol{r}', t') \tag{C.207}$$

in a convolution relation, which is consistent because of

$$\Box\psi(\boldsymbol{r}, t) = \int dV' \int dt'\, \Box G(\boldsymbol{r} - \boldsymbol{r}', t, -t')q(\boldsymbol{r}', t') = \\ 4\pi \int dV' \int dt' \delta_D(\boldsymbol{r} - \boldsymbol{r}')\delta_D(t - t')q(\boldsymbol{r}', t') = 4\pi q(\boldsymbol{r}, t) \tag{C.208}$$

as a consequence of the shifting relation of the δ_D-function.

In Fourier-space, the Green-function is given by

$$G(\omega, \boldsymbol{k}) = \int dV \int dt\, G(\boldsymbol{r} - \boldsymbol{r}', t - t')\exp(-i\omega(t - t'))\exp(-ik_i(\boldsymbol{r} - \boldsymbol{r}')^i) \tag{C.209}$$

with the inversion

$$G(\boldsymbol{r} - \boldsymbol{r}', t - t') = \frac{1}{(2\pi)^4}\int d\omega \int d^3k\, G(\omega, \boldsymbol{k})\exp(i\omega(t - t'))\exp(ik_i(\boldsymbol{r} - \boldsymbol{r}')^i) \tag{C.210}$$

so that the application of the d'Alembert-operator gives

$$\Box G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \Box\frac{1}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}^3k\, G(\omega,\boldsymbol{k})\exp(\mathrm{i}\omega(t-t'))\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i) =$$

$$-\frac{1}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}^3k \left[\frac{\omega^2}{c^2}-k^2\right] G(\omega,\boldsymbol{k})\exp(\mathrm{i}\omega(t-t'))\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i), \quad \text{(C.211)}$$

as ∂_{ct} acts on $\exp(\mathrm{i}\omega(t-t'))$, and ∂_a on $\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i)$, yielding $\mathrm{i}\omega/c$ and $\mathrm{i}k_a$ each twice; and we abbreviate $k^2 = \gamma^{ab}k_ak_b$:

$$\begin{array}{ccc} \psi(\boldsymbol{r},t) & \xleftarrow{\mathcal{F}^{-1}} & \psi(\boldsymbol{k},\omega) \\ \Box\downarrow & & \downarrow \omega^2/c^2-\gamma^{ij}k_ik_j \\ q(\boldsymbol{r},t) & \xrightarrow{\mathcal{F}} & q(\boldsymbol{k},\omega) \end{array} \quad \text{(C.212)}$$

On the other hand, this expression needs to be equal, according to eqn. (C.206), to the Fourier-representation of the δ_D-distributions,

$$\delta_D(t-t')\delta_D(\boldsymbol{r}-\boldsymbol{r}') = \frac{1}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}^3k\; \exp(\mathrm{i}\omega(t-t'))\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i) \quad \text{(C.213)}$$

with all frequencies appearing at equal amplitude. By comparing the latter two expressions, one can extract the Fourier-transformed Green-function $G(\omega,\boldsymbol{k})$ to be

$$G(\omega,\boldsymbol{k}) = 4\pi\frac{c^2}{\omega^2-(ck)^2}. \quad \text{(C.214)}$$

But transforming back to configuration space reveals a problem: Formally, one writes down

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{1}{4\pi^3}\int \mathrm{d}\omega \int \mathrm{d}^3k\, \frac{c^2}{\omega^2-(ck)^2}\exp(\mathrm{i}\omega(t-t'))\exp(\mathrm{i}k_i(\boldsymbol{r}-\boldsymbol{r}')^i) \quad \text{(C.215)}$$

where one encounters two singularities of the integrand at $\omega = \pm ck$ when performing the $\mathrm{d}\omega$-integration, for every value of k, as indicated by Fig. 7. This issue is most elegantly solved by the methods of complex integration.

For carrying out the $\mathrm{d}\omega$-integration one can extend the function to complex arguments and close the integration path along the real axis by a loop: In this way, one deals with a closed loop integral over a holomorphic function, where the two poles can be shifted inside the integration contour by adding $+\mathrm{i}\epsilon$ to them, which does not change the final result. Then, the value of the integral is entirely fixed by the values of the two residuals associated with the two poles:

$$\int \mathrm{d}\omega\, \frac{c^2}{(ck)^2-\omega^2}\exp(\mathrm{i}\omega(t-t')) \to -c^2\oint \mathrm{d}\omega\, \frac{\exp(\mathrm{i}\omega(t-t'))}{\omega^2-(ck)^2}, \quad \text{(C.216)}$$

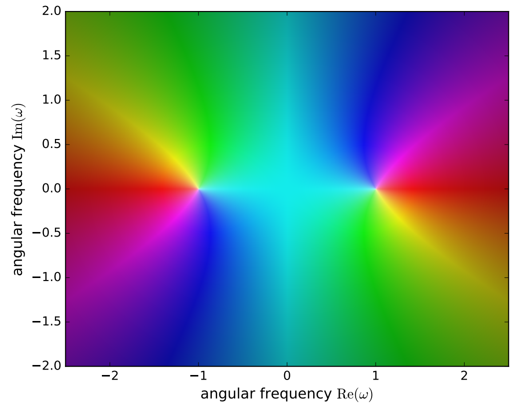


Figure 7: Function $1/(\omega^2 - (ck)^2)$ *over the complex plane* $\omega = \mathrm{Re}(\omega) + \mathrm{i}\,\mathrm{Im}(\omega)$, *for* $ck = 1$, *with color indicating phase and hue indicating the absolute value. The two singularities at* $\omega = \pm ck$ *are clearly visible.*

where the denominator factorises $(\omega^2 - (ck)^2) = (\omega + ck)(\omega - ck)$, by virtue of the binomial formula.

Let's investigate the residues at the two poles at $\omega_+ = \omega + ck$ and $\omega_- \omega - ck$ separately: Computing the residues requires the limits

$$\mathrm{Res}_+ = \lim_{\omega \to +ck} (\omega - ck) \frac{\exp(\mathrm{i}\omega(t-t'))}{(\omega + ck)(\omega - ck)} = -\frac{c}{2k}\exp(+\mathrm{i}ck(t-t')) \tag{C.217}$$

and

$$\mathrm{Res}_- = \lim_{\omega \to -ck} (\omega + ck) \frac{\exp(\mathrm{i}\omega(t-t'))}{(\omega + ck)(\omega - ck)} = +\frac{c}{2k}\exp(-\mathrm{i}ck(t-t')) \tag{C.218}$$

Cauchy's ✈ residue theorem now states that the value of the loop integral is equal to the sum of the residues, up to a factor of $2\pi\mathrm{i}$,

$$\oint \mathrm{d}\omega \, \frac{\exp(\mathrm{i}\omega(t-t'))}{\omega^2 - (ck)^2} = 2\pi\mathrm{i}\,(\mathrm{Res}_+ + \mathrm{Res}_-) =$$
$$\frac{2\pi}{\mathrm{i}}\left(\frac{c}{2k}\exp(+\mathrm{i}ck(t-t')) - \frac{c}{2k}\exp(-\mathrm{i}ck(t-t'))\right) = \frac{2\pi c}{k}\sin(ck(t-t')) \tag{C.219}$$

The remaining d^3k-integration reads:

$$G(\boldsymbol{r} - \boldsymbol{r}', t - t') = \frac{c}{2\pi^2}\int \mathrm{d}^3k \, \frac{\sin(ck(t-t'))}{k}\exp(\mathrm{i}k_i(\boldsymbol{r} - \boldsymbol{r}')^i) \tag{C.220}$$

and can be most sensibly carried out in spherical coordinates: $\mathrm{d}^3k = k^2\mathrm{d}k\mathrm{d}\mu\mathrm{d}\phi$, with azimuthal symmetry and μ being the cosine of the angle between $\boldsymbol{k}$ and $\boldsymbol{r} - \boldsymbol{r}'$.

Then,

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{c}{\pi}\int_0^\infty k\mathrm{d}k\ \sin(ck(t-t'))\int_{-1}^{+1}\mathrm{d}\mu\ \exp(\mathrm{i}\mu k\left|-\boldsymbol{r}-\boldsymbol{r}'\right|) \tag{C.221}$$

The dμ-integral has an elementary solution in term of the sine, so we arrive at

$$G(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\frac{2c}{\pi}\int_0^\infty \mathrm{d}k\ \sin(ck(t-t'))\sin(k\left|\boldsymbol{r}-\boldsymbol{r}'\right|) \tag{C.222}$$

The integral can be carried out by rewriting both sines as differences of complex exponentials, multiplying out the expression and integrate. For convenience, we abbreviate $\Delta t = t - t'$ and $\Delta r = |\boldsymbol{r}-\boldsymbol{r}'|$:

$$2\int_0^\infty \mathrm{d}k\ \sin(ck\Delta t)\sin(k\Delta r) = \int_{-\infty}^{+\infty}\mathrm{d}k\ \sin(ck\Delta r)\sin(k\Delta r) =$$

$$\frac{1}{(2\mathrm{i})^2}\int_{-\infty}^{+\infty}\mathrm{d}k\ \left(\exp\left(+\mathrm{i}ck\Delta t\right)-\exp\left(-\mathrm{i}ck\Delta t\right)\right)\times\left(\exp\left(+\mathrm{i}k\Delta r\right)-\exp\left(-\mathrm{i}k\Delta r\right)\right). \tag{C.223}$$

Rearranging the terms leads to

$$\ldots = \frac{1}{(2\mathrm{i})^2}\int_{-\infty}^{+\infty}\mathrm{d}k\ \exp(+\mathrm{i}k[c\Delta t+\Delta r]) + \exp(-\mathrm{i}k[c\Delta t+\Delta r]) -$$

$$\exp(+\mathrm{i}k[c\Delta t-\Delta r]) - \exp(-\mathrm{i}k[c\Delta t-\Delta r]), \tag{C.224}$$

where one recognises the sum and difference of the two frequencies $c\Delta t$ and Δr. The integrals are effectively the Fourier-representation of the δ_D-function,

$$\int_{-\infty}^{+\infty}\mathrm{d}k\ \exp(\mathrm{i}kx) = 2\pi\delta_D(x) \tag{C.225}$$

so that one arrives at

$$\ldots = \frac{4\pi}{(2\mathrm{i})^2}\delta_D(c\Delta t+\Delta r) - 4\pi\delta_D(c\Delta t-\Delta r) \tag{C.226}$$

as each term appears twice. By applying the scaling property of Dirac's δ_D-function

$$\delta_D(\alpha k) = \frac{1}{\alpha}\delta_D(k) \tag{C.227}$$

one arrives at

$$\ldots = -\frac{4\pi}{c}\delta_D\left(t - t' + \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) + \frac{4\pi}{c}\delta_D\left(t - t' - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \tag{C.228}$$

where the factors c and π cancel with the corresponding factors in eqn. (C.222). Putting everything together yields as a final result for the Green-function

$$G_\pm(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\Big[\underbrace{\delta_D(t - t' - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c})}_{\text{retarded}} - \underbrace{\delta_D(t - t' + \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c})}_{\text{advanced}}\Big] \tag{C.229}$$

with the conventional Green-function of Δ as a prefactor, modified by δ_D-functions. They take care of the fact that changes in the fields propagate at finite speed, such that the source configuration at distance $|\boldsymbol{r}-\boldsymbol{r}'|$ contributes to the potential at most at a time $|\boldsymbol{r}-\boldsymbol{r}'|/c$ *later* than t', which necessitates that one of the terms is discarded as being acausal: It would have the effect, that a source configuration at a time difference $|\boldsymbol{r}-\boldsymbol{r}'|/c$ in the *future* contributes to the fields. Finally, one arrives at the expression for the *retarded* Green-function $G_-(\boldsymbol{r}-\boldsymbol{r}', t-t')$,

$$G_-(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\delta_D\left(t - t' - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right), \tag{C.230}$$

which serves for determining the potential $\psi(\boldsymbol{r}, t)$ from the source $q(\boldsymbol{r}', t')$,

$$\psi(\boldsymbol{r}, t) = \int \mathrm{d}V' \int \mathrm{d}t' \, \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\delta_D\left(t - t' - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) q(\boldsymbol{r}', t') = \int \mathrm{d}V' \, \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} q\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right). \tag{C.231}$$

C.8 *Liénard-Wiechert potentials*

With the Green-functions for the d'Alembert-operator $\Box$,

$$G_\pm(\boldsymbol{r}-\boldsymbol{r}', t-t') = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\delta_D\left(t - t' \pm \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \tag{C.232}$$

it is possible to solve the wave equation

$$\Box\psi(\boldsymbol{r}, t) = 4\pi q(\boldsymbol{r}, t) \tag{C.233}$$

in a convolution relation,

$$\psi_\pm(\boldsymbol{r}, t) = \int \mathrm{d}t' \int \mathrm{d}V' \, G_\pm(\boldsymbol{r}-\boldsymbol{r}', t-t') q(\boldsymbol{r}', t') \tag{C.234}$$

where changes to the source configuration $q(\boldsymbol{r}', t')$ (to be interpreted as the charge distribution $\rho(\boldsymbol{r}, t)$ or the current density $\gamma_{ij}\jmath^i(\boldsymbol{r}, t)$) can only influence the fields (or potentials $\Phi(\boldsymbol{r}, t)$ and $A_i(\boldsymbol{r}, t)$, even though this statement depends on the gauge choice) after a time $|\boldsymbol{r}-\boldsymbol{r}'|/c$ has elapsed, and not instantaneously, due to the finite propagation speed c of excitations in the electromagnetic field.

Substituting the retarded Green-function G_- into the convolution relation for the potentials for obtaining them from the source distribution one arrives at

$$\psi(\boldsymbol{r},t) = \int \mathrm{d}V' \int \mathrm{d}t'\, G_-(\boldsymbol{r}-\boldsymbol{r}', t-t') q(\boldsymbol{r}', t') =$$

$$\int \mathrm{d}V' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \int \mathrm{d}t'\, \delta_D\left(t - t' - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) q(\boldsymbol{r}', t') = \int \mathrm{d}V' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} q\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \tag{C.235}$$

because the Dirac-δ_D fixes t' to the value $t - |\boldsymbol{r}-\boldsymbol{r}'|/c$. This expression applied to the potentials

$$\Phi(\boldsymbol{r},t) = \int \mathrm{d}V' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \rho\left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \tag{C.236}$$

and

$$A_i(\boldsymbol{r},t) = \int \mathrm{d}V' \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \gamma_{ij} \jmath^j \left(\boldsymbol{r}', t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right) \tag{C.237}$$

is referred to as the Liénard-Wiechert potentials, which provide a solution in the case a time-varying source distribution, taking retardation, i.e. the finite speed of propagation of the fields (or potentials in Lorenz gauge) into account. Clearly, in the limit $c \to \infty$ the fields and potentials would change instantaneously. Already now a causal structure becomes apparent, with a finite propagation speed at which the fields react to changes in the source. Taking the derivatives $B^i = \epsilon^{ijk}\partial_j A_k$ and $E_i = -\partial_i \Phi - \partial_{ct} A_i$ then leads to Jefimenko's equations, if one interchanges differentiation ∂_i and ∂_{ct} with the $\mathrm{d}V'$-integration for an expression for the fields for the case of time varying sources.

C.9 *Anatomy of partial differential equations*

Differential equations are the natural language in which laws of Nature are formulated: They set the rates of change of quantities into relation and depend crucially on initial and boundary conditions. Many different categories are relevant in the classification of differential equations:

- ordinary vs. partial:

  In ordinary differential equations, only derivatives with respect to a single variable or coordinate appear, whereas partial differential equations consist of derivatives with respect to two or more variables.

- homogeneous vs. inhomogeneous:

  If all terms depend on the field and its derivatives, the differential equation is homogeneous, but if a term appears that does not depend on the field or its derivatives, the equation is inhomogeneous.

- linear vs. nonlinear:

  If all terms in a differential equation are proportional to the field or its derivatives, the equation is linear, but if there are higher-order powers or nonlinear functions of the field, then the differential equation is nonlinear.

- derivative order:

  The highest derivative that appears in the differential equation sets the derivative order.

Given these definitions, the damped harmonic oscillator equation for the amplitude $x(t)$ with external driving $a(t)$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x(t) = a(t) \tag{C.238}$$

is an ordinary, inhomogeneous, linear differential equation of second order. The Schrödinger equation

$$\mathrm{i}\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + \Phi(r)\psi \tag{C.239}$$

on the other hand is a partial, homogeneous and linear differential equation, but its derivative order is likewise two.

It's well worth going through this categorisation as a checklist whenever you need to deal with ODEs/PDEs.

C.9.1 Hyperbolic, parabolic and elliptical differential equations

We have already encountered two partial differential equations of second order, the Laplace-equation

$$\Delta\Phi = \gamma^{ij}\partial_i\partial_j\Phi = 0 \tag{C.240}$$

as the field equation of electrostatics, and the wave equation

$$\Box\Phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = \left(\partial_{ct}^2 - \Delta\right)\Phi = 0 \tag{C.241}$$

of electrodynamics, here obtained in Lorenz-gauge. It suffices to consider the case of homogeneous partial differential equations because any inhomogeneity $\pm 4\pi\rho$ could be dealt with the Green-formalism. Comparing $\Box\Phi = 0$ as a wave equation with $\Delta\Phi = 0$ as a static field equation shows that the signs of the derivative operators $(+,-,-,-)$ and $(+,+,+)$ matter a lot, as one obtains oscillatory solutions for the wave equation, and (decreasing, at least in 3 dimensions or more) power-law solutions for the Poisson-equation. Please note that the choice of gauge does not have any influence at all on the derivative order (it is a statement involving only the first derivatives of the fields), but that it can change the character between hyperbolical and elliptical.

The classification of differential equations borrows many ideas from curves, here in particular from the theory of conic sections. A quadratic form of two coordinates x and y would be given by

$$\begin{pmatrix} x \\ y \end{pmatrix}^t \underbrace{\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}}_{=\mathrm{D}} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + bxy + cy^2 = \text{const.} \tag{C.242}$$

Depending on the structure of eigenvalues, which decide on the sign of the determinant of the (discriminant) matrix D, the quadratic form describes very different curves: If $b = 0$ (for simplicity) and $a = c = 1 > 0$ one obtains $x^2 + y^2 = \text{const}$, which can be rewritten in a parametric form by setting $x = \cos\varphi$ and $y = \sin\varphi$ such that the quadratic form describes a circle as a consequence of $\cos^2\varphi + \sin^2\varphi = 1$, and in the peculiar case of $a \neq c$ an ellipse. If $a = 1$ and $c = -1$, the quadratic form becomes $x^2 - y^2 = \text{const}$, i.e a hyperbola with the hyperbolic functions as parametric forms, using $\cosh^2\psi - \sinh^2\psi = 1$. More generally, the picture arises that $\det \mathrm{D} > 0$ for the elliptical conic section and conversely, $\det \mathrm{D} < 0$ for the hyperbolic conic section.

Applying this idea to the classification of partial differential equations, we start with a homogeneous second-order PDE for the field ϕ in two coordinates in full generality,

$$a(x,y)\frac{\partial^2}{\partial x^2}\phi(x,y) + b(x,y)\frac{\partial^2}{\partial x\partial y}\phi(x,y) + c(x,y)\frac{\partial^2}{\partial y^2}\phi(x,y) = \mathrm{A}(x,y)\phi(x,y) \quad \text{(C.243)}$$

and assemble the matrix D

$$\mathrm{D} = \begin{pmatrix} a(x,y) & \frac{1}{2}b(x,y) \\ \frac{1}{2}b(x,y) & c(x,y) \end{pmatrix} \quad \text{(C.244)}$$

The determinant of D then establishes, whether the PDE is elliptical, $\det \mathrm{D} > 0$, parabolic, $\det \mathrm{D} = 0$ or hyperbolic, $\det \mathrm{D} < 0$. A visual impression is provided by Fig. 8 which shows these curves, actually conic sections, for various choices of the parameters.

Sticking to 2 dimensions, a PDE like the Poisson-equation

$$\Delta\phi = \frac{\partial^2}{\partial x^2}\phi(x,y) + \frac{\partial^2}{\partial y^2}\phi(x,y) = 0 \quad \text{(C.245)}$$

would be elliptical, as the determinant of D would come out positive: $a = c = 1$ and $b = 0$: elliptical differential equations have only unique solutions after boundary conditions are specified. They can be of the Dirichlet-type, the Neumann-type or be of mixed type. Please note that vacuum boundary conditions, where the fields and their derivatives approach zero at infinity, are perfectly admissible. Typical solutions are decreasing (for Poisson-like problems, at least in 3 dimensions or higher) with increasing coordinates and parity invariant, as $(x,y) \to (-x,-y)$ does not change anything.

On the other hand, a wave-equation exhibits a sign change,

$$\Box\phi(t,x) = \frac{\partial^2}{\partial(ct)^2}\phi(t,x) - \frac{\partial^2}{\partial x^2}\phi(t,x) = 0 \quad \text{(C.246)}$$

with $a = 1$, $c = -1$ and $b = 0$ in these coordinates and would be hyperbolic as $\det \mathrm{D} < 0$. In this case, it is enough to specify initial conditions and the PDE evolves them in a well-defined and unique way into the future. Specification of boundary conditions as in the case of elliptical PDEs is unnecessary, and in contrast to elliptical PDEs, hyperbolic PDEs show typically wavelike-solutions.

There is clearly the notion of a light-cone due to retardation, which persists even when a change of coordinates is carried out: Switching to light-cone coordinates $\partial_u = \partial_{ct} + \partial_x$ and $\partial_v = \partial_{ct} - \partial_x$ brings the wave equation into the form

$$\Box\phi(u,v) = \frac{\partial^2}{\partial u \partial v}\phi(u,v) = 0 \quad \text{(C.247)}$$

⚠ *Please go through all iconic PDEs in theoretical physics and classify them as elliptical, parabolic or hyperbolic partial differential equations!*

this time with $a = c = 0$ and $b = 1$, but the determinant $\det D < 0$ nonetheless. It is actually the case that the metric structure of spacetime, which we focus on in the next chapter, with the Minkowski-metric is uniquely suited for hyperbolic PDEs: It is even the fact. The Lorentzian spacetime is the only metric spacetime with naturally hyperbolic evolution!

C.9.2 Wave-equation and its reductions

Central to electrodynamic theory was the wave-equation

$$\Box\phi(\boldsymbol{r}, t) = 4\pi q(\boldsymbol{r}, t) \quad \text{with} \quad \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_{ct}^2 - \Delta \quad \text{and} \quad \Delta = \gamma^{ij}\partial_i\partial_j \tag{C.248}$$

as a linear, inhomogeneous, hyperbolical, partial differential equation of derivative order two. Separating out oscillations in time with an ansatz $\phi \propto \exp(\pm i\omega t)$ leads to the ✈ Helmholtz-equation

$$\Delta\phi + k^2\phi = -4\pi q(\boldsymbol{r}, t) \tag{C.249}$$

with $k = \omega/c$. Under the stronger assumption of a static solution, where neither ϕ nor q depended on t, one arrives at the Poisson-equation,

$$\Delta\phi = -4\pi q(\boldsymbol{r}) \tag{C.250}$$

further reducing to the Laplace-equation

$$\Delta\phi = 0 \tag{C.251}$$

for the vacuum case with vanishing sources. In all cases is the incorporation of an inhomogeneity $q(\boldsymbol{r}, t)$ straightforwardly possible by means of the Green-formalism.

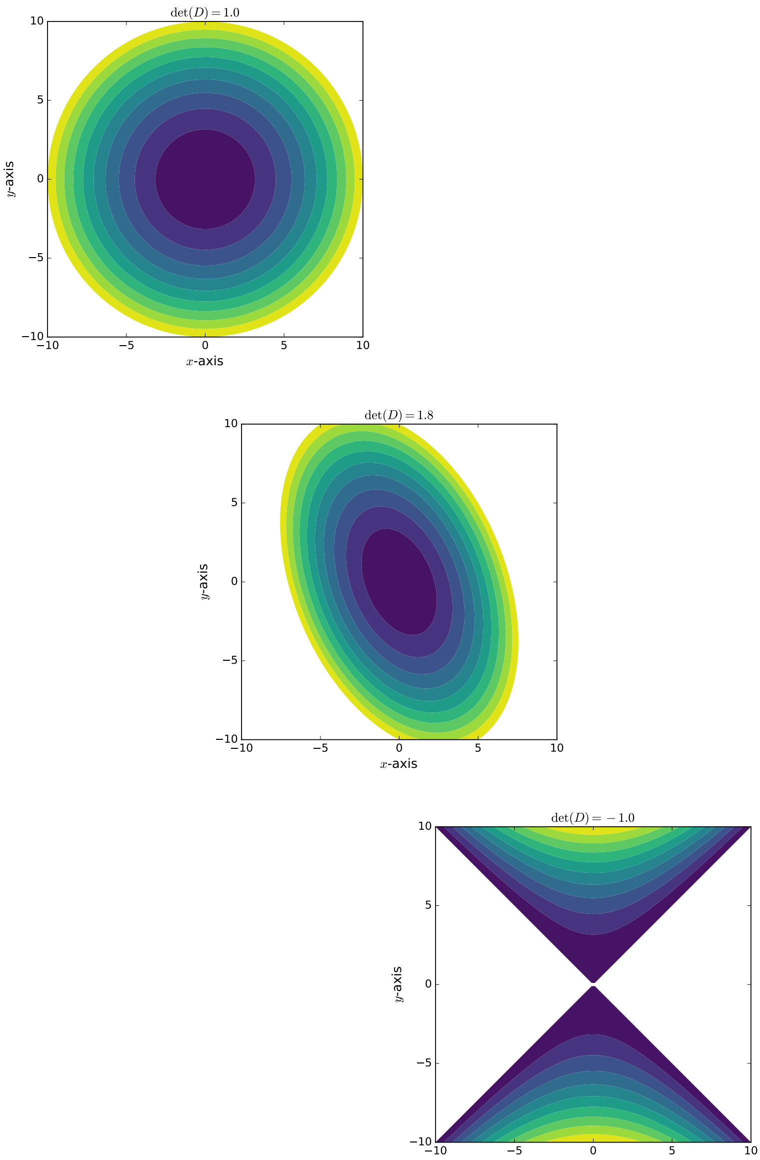


Figure 8: Conic sections: circles ($\det(\mathrm{D}) = 1$*), ellipses (*$\det \mathrm{D} > 0$*) and hyperbolæ(*$\det(\mathrm{D}) < 0$*), from top to bottom.*

D SPECIAL RELATIVITY

D.1 *Lorentz-transforms*

The relativity principle stipulates that the laws of Nature and the constants of Nature should be the same in all frames, or in other words: There is no preferred frame in which the laws of Nature should be formulated. Space, or spacetime is homogeneous as there neither a particular location nor a particular instant in time for the formulation of laws of Nature, and the transition between one coordinate choice and the next one should be a linear, affine function: Any nonlinearity would single out a particular location or instant, breaking homogeneity. In short, the transition between frames S and S′, with their associated coordinates x^μ and x'^μ,

$$\mathrm{S} : x^\mu = \begin{pmatrix} t \\ x^i \end{pmatrix} \quad \rightarrow \quad \mathrm{S}' : x^\mu = \begin{pmatrix} t' \\ x'^i \end{pmatrix} \tag{D.252}$$

is necessarily an affine transformation.

There is a very good physical argument why this needs to be the case: Imagine now that an observer with a clock moves through spacetime on a trajectory with coordinates $x^\mu\tau$ as seen by S, and coordinates $x'^\mu(\tau)$ as seen by S′, where the parameter τ by which the trajectory is parameterised, is the proper time of the observer - the time displayed on her or his wrist watch. For an inertial trajectory, where all accelerations are zero, the velocity $\upsilon^i = \mathrm{d}x^i/\mathrm{d}\tau$ is constant, as well as the size of the time intervals $\mathrm{d}t/\mathrm{d}\tau$. In summary,

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \mathrm{const} \quad \text{and} \quad \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} = 0 \tag{D.253}$$

But that statement needs to be true within the frame S′ just as well:

$$\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} = \mathrm{const} \quad \text{and} \quad \frac{\mathrm{d}^2 x'^\mu}{\mathrm{d}\tau^2} = 0 \tag{D.254}$$

Coordinate transforms can be written as an invertible, and differentiable functional relationship between the coordinate sets, i.e. in the form $x'(x)$. In this case, the velocity in the new coordinate choice becomes

$$\frac{\mathrm{d}x'^\mu}{\mathrm{d}\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{D.255}$$

with a Jacobian $\partial x'^\mu/\partial x^\nu$ mediating the coordinate change. The acceleration though acquires two terms, as both the Jacobian as well as the velocity could change with τ, albeit indirectly through the trajectory $x^\mu(\tau)$:

$$\frac{\mathrm{d}^2 x'^\mu}{\mathrm{d}\tau^2} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} + \underbrace{\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho}}_{\mathrm{A}^\mu{}_{\nu\rho}}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \tag{D.256}$$

Only if the term $\mathrm{A}^\mu{}_{\nu\rho}$ is equal to zero, one can conclude from $\mathrm{d}^2x^\mu/\mathrm{d}\tau^2 = 0$ that $\mathrm{d}^2x'^\mu/\mathrm{d}\tau^2 = 0$. But then, the transformation between the two coordinate frames is linear:

$$\frac{\partial^2 x'^\mu}{\partial x^\nu \partial x^\rho} = 0 \quad \rightarrow \quad \frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu \quad \rightarrow \quad x'^\mu = \Lambda^\mu{}_\nu x^\mu + b^\mu \tag{D.257}$$

with integration constants $\Lambda^\mu{}_\nu$ and b^μ.

To be more specific one needs some empirical, physical input: Let's assume that the two frames S and S′ move at a constant relative speed υ. Without loss of generality, the two frames should be oriented in the same direction and the relative displacement should be along the x-axis of the coordinate frame, and the two frames should coincide in their origins at $t = t' = 0$. Then, the origin of S has the coordinate $x' = -\upsilon t'$ seen from S′, whereas the origin of S′ is at $x = +\upsilon t$ from the point of view of S.

Linearity of the transforms commands that $x' = ax + bt$ with two constant coefficients a and b, that can be functions of υ. Because $x = \upsilon t$ implies $x' = 0$, one can write: $x' = 0 = a\upsilon t + bt = (a\upsilon + b)t$, from which follows that $b = -a\upsilon$ and therefore $x' = a(x - \upsilon t)$. Reversing the roles of S and S′ then requires from $x = ax' + bt'$ that $x' = -\upsilon t$ if $x = 0$ should hold, implying $x = 0 = -a\upsilon x' + bt' = (-a\upsilon + b)t'$, and consequently $b = +a\upsilon$ and $x = a(x' + \upsilon t')$. The symmetry of the transform has effectively reduced the number of free parameters from two to a single one.

At this point Nature can make a choice. Most straightforwardly, she might choose the time to be universal, $t = t'$, and humans thought this was the case until 1905. $x' = a(x - \upsilon t)$ and $x = a(x' + \upsilon t)$ can only be compatible if $a = 1$, leading us straight to the Galilei-transforms. Or, the speed of light could be the same in all frames, $c = c'$, with $x = ct$ in S and $x' = ct'$ in S′, as the distance a light signal covers in the two respective frames. Then,

$$\begin{cases} ct &= a(ct' + \upsilon t') = a(c + \upsilon)t' \\ ct' &= a(ct - \upsilon t) = a(c - \upsilon)t \end{cases} \tag{D.258}$$

Multiplying both equations leads to $c^2 tt' = a^2(c + \upsilon)(c - \upsilon)tt'$, such that

$$a \equiv \gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad \text{with} \quad \beta = \frac{\upsilon}{c}. \tag{D.259}$$

The quantity γ is known as the Lorentz-factor, and by convenience one works with a dimensionless velocity $\beta = \upsilon/c$.

A Taylor-expansion for small velocities β, $|\beta| \ll 1$, or $|\upsilon| \ll c$ of the Lorentz-factor yields

$$\gamma = 1 + \left.\frac{d\gamma}{d\beta}\right|_{\beta=0} \beta + \left.\frac{d^2\gamma}{d\beta^2}\right|_{\beta=0} \frac{\beta^2}{2} + \cdots = 1 + \frac{\beta^2}{2} + \cdots \tag{D.260}$$

showing that the Lorentz-factor depends to lowest order quadratically on the velocity before diverging as β approaches unity.

The definition of $\beta = \upsilon/c$ allows a more consistent notation for Lorentz transforms: ct as a time coordinate is then measured in units of length, just as x, there is no ambiguity as c has by virtue of the relativity principle the same value in all frames. The term υt in the Lorentz transform becomes βct, leading to

$$\begin{cases} ct' &= \gamma(ct + \beta x) \\ x' &= \gamma(x + \beta ct) \end{cases} \tag{D.261}$$

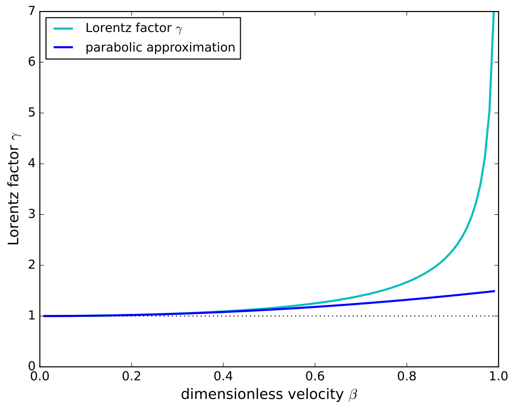


Figure 9: Lorentz γ-factor as a function of dimensionless velocity β, and the parabolic approximation for small β.

Alternatively, the transformation reads in matrix notation,

$$\underbrace{\begin{pmatrix} ct' \\ x' \end{pmatrix}}_{x'^{\mu}} = \underbrace{\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix}}_{\Lambda^{\mu}{}_{\nu}} \underbrace{\begin{pmatrix} ct \\ x \end{pmatrix}}_{x^{\nu}} \tag{D.262}$$

with a clearly common transformation of the ct and x coordinates, that are now combined into a single vector x^{μ}, following the transformation law $x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$. For small velocities, $\gamma \simeq 1$ and one obtains

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{D.263}$$

With either positive or negative off-diagonal elements it is clear that a coordinate frame undergoes a shearing under Lorentz transforms, in contrast to antisymmetric transformation matrices in the case of rotations. Quantitatively for small velocities $\upsilon \ll c$ the relation reduces to $t' = t$ (neglecting $\beta x = \upsilon x/c$ for $\upsilon \ll c$) and $x' = \upsilon t + x$ in recovery of the Galilei transform.

D.2 *Lorentz-invariants*

While the coordinates depend on a chosen frame and undergo a joint change under Lorentz tranforms, one might wonder whether there are quantities that remain constant and offer the possibility to say something true for a system that would not depend on the choice of frame. Clearly, rotations leave the length of a vector, defined as its norm $r^2 = \delta_{ij} x^i x^j$ unchanged, and in this vein one can construct the quantity

$$(ct')^2 - (x')^2 = \gamma^2 \left((ct)^2 - 2ct\beta x + \beta^2 x^2 - x^2 + 2ct\beta x - \beta^2 (ct)^2\right) = \underbrace{\gamma^2(1-\beta^2)}_{=1}\left((ct)^2 - x^2\right) \tag{D.264}$$

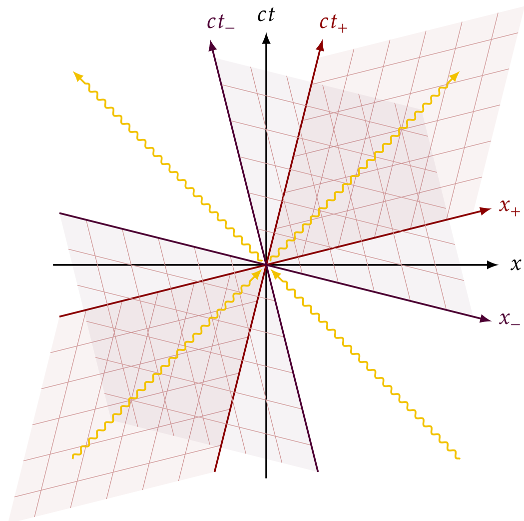


Figure 10: Spacetime diagrams under Lorentz transform for positive velocities (ct_+, x_+) *and negative velocities* (ct_-, x_-) *relative to the frame* (ct, x)*. Reproduction with kind permission of I. Neutelings.*

which remains in fact constant under Lorentz transforms. In order to write the invariant quantity $s^2 = (ct)^2 - \delta_{ij}x^i x^j$, extended to three spatial dimensions, one introduces the Minkowski-metric,

$$s^2 = \eta_{\mu\nu}x^\nu x^\nu \quad \text{with} \quad \eta_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{D.265}$$

which combines the Euclidean scalar product $r^2 = \gamma_{ij}x^i x^j$ mediated by by the Euclidean metric γ_{ij} to the new invariant $s^2 = \eta_{\mu\nu}x^\mu x^\nu$, as soon as Lorentz boosts are involved.

D.3 *Rapidity*

Rotations of the coordinate frame can be written in terms of a rotation matrix,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{D.266}$$

which begs the question whether (i) a similar parameterisation of the group of Lorentz transforms is possible, and if yes, (ii) which parameter ψ would replace the rotation angle α. A Lorentz-boost, written in matrix notation, would be

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{D.267}$$

where a notable structural difference is of course the different sign in the lower left corner. But clearly, we are not looking for a rotation, as the Lorentz invariant s^2 differs from the invariant r^2! The values of the entries of the matrix are $1 \leq \gamma < +\infty$ as well as $-\infty < \beta\gamma < +\infty$, which an additional symmetry of γ for positive and negative velocities, and a sign change of $\beta\gamma$. With a bit of intuition, one might be tempted to use the hyperbolic functions to set $\gamma = \cosh\psi$ and $\beta\gamma = \sinh\psi$ (compare Fig. 11) with the so-called rapidity ψ,

$$\tanh\psi = \frac{\sinh\psi}{\cosh\psi} = \frac{\beta\gamma}{\gamma} = \beta \quad \to \quad \psi = \operatorname{artanh}\beta = \frac{1}{2}\ln\frac{1+\beta}{1-\beta}. \tag{D.268}$$

where the inverse hyperbolic tangent has a surprising representation in terms of elementary functions. More accurately, one might use the relation $\gamma^2(1+\beta^2) = 1$ to verify that

$$\gamma^2(1-\beta^2) = \gamma^2 - \gamma^2\beta^2 = \cosh^2\psi - \sinh^2\psi = 1 \tag{D.269}$$

as the defining characteristic of the hyperbolic functions.

The rapidity ψ diverges as $\beta \to 1$ and keeps, due to the antisymmetry of the hyperbolic sine, information about the direction of the boost velocity. With the rapidity as a parameter, the Lorentz-boost can be written as a hyperbolic "rotation",

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}. \tag{D.270}$$

Then, the invariant $s^2 = (ct)^2 - x^2$ is unchanged because $\cosh^2\psi - \sinh^2\psi = \gamma^2 - \beta^2\gamma^2 = \gamma^2(1-\beta^2) = 1$, just as the invariant $r^2 = x^2 + y^2$ is unchanged because of $\cos^2\alpha + \sin^2\alpha = 1$. For a more geometric intuition, one can imagine that any point (ct, x) follows a hyperbola, purely in the timelike region for a positive norm or in the spacelike region of the spacetime diagram in the case of a negative norm. Along these hyperbolae, the norm is strictly conserved. Taking things to extremes would be a point with a lightlike norm $s^2 = 0$, which moves along the diagonals of the spacetime diagram.

D.4 *Spacetime symmetries*

A notion of spacetime was established fusion of the spatial and temporal coordinates into a coordinate tuple x^μ and the extension of the Euclidean scalar product $x_i y^i = \gamma_{ij} x^i y^j$ to the Minkowski scalar product $x_\mu y^\mu = \eta_{\mu\nu} x^\mu y^\nu$. Lorentz-transforms and rotations act on these coordinate tuples, $x^\mu \to \Lambda^\mu{}_\alpha x^\alpha$ and $x^i \to \mathrm{R}^i{}_a x^a$, respectively, leaving the scalar products invariant, $\eta_{\mu\nu} x^\mu x^\nu \to \eta_{\mu\nu} \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta x^\alpha x^\beta = \eta_{\alpha\beta} x^\alpha x^\beta$ and $\gamma_{ij} x^i x^j \to \gamma_{ij} \mathrm{R}^i{}_a \mathrm{R}^j{}_b x^a x^b = \gamma_{ab} x^a x^b$, expressed in coordinates $s^2 = \eta_{\mu\nu} x^\mu x^\nu = (ct)^2 - x^2 - y^2 - z^2$ and $r^2 = \gamma_{ij} x^i x^j = x^2 + y^2 + z^2$.

Clearly, the Lorentz-transforms as well as the rotations form groups: Successive transforms can be summarised into a single transform, for each transform there is an inverse (boosting with the negative velocity and rotating by a negative angle), and the neutral element is part of each group (corresponding to a boost with velocity zero or a rotation by an angle of zero). But there seems to be a peculiarity: The groups contain uncountably many elements and are parameterised by a continuous, real valued parameter (rapidity ψ or rotation angle α). As such, they are examples of Lie-groups.

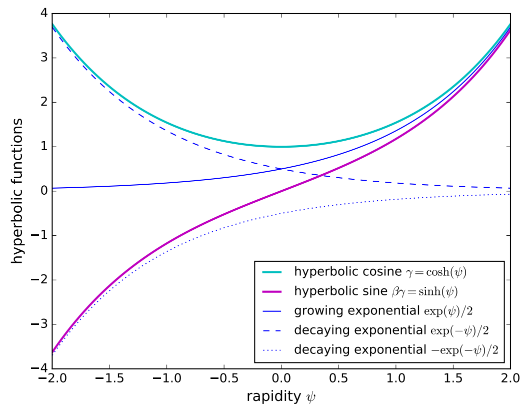


Figure 11: Hyperbolic functions $\gamma = \cosh(\psi)$ *and* $\beta\gamma = \sinh(\psi)$ *with exponentials as their asymptotics, as a function of the rapidity* ψ.

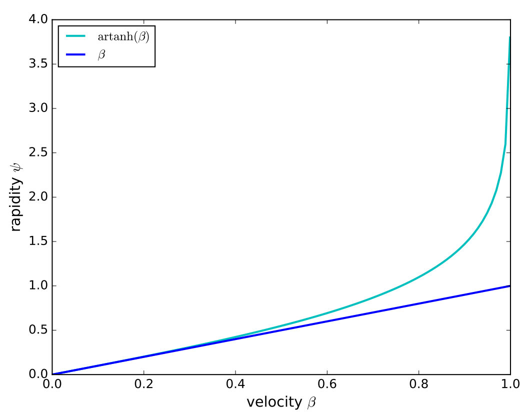


Figure 12: Rapidity ψ *as a function of velocity* β *with the clear divergence as* $\beta \to 1$.

Because of the real-valued parameter, one can actually perform a differentiation of the group element with respect to that parameter, consider an infinitesimal transform and assemble all possible group elements from this infinitesimal transform as a building block:

In the case of rotations in 2 dimensions by a small angle α one could expand the rotation matrix $R^i{}_a(\alpha)$ into a Taylor-series,

$$R^i{}_a(\alpha) = R^i{}_a\big|_{\alpha=0} + \frac{d}{d\alpha}R^i{}_a\big|_{\alpha=0}\,\alpha = \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma^{(0)} + \alpha\sigma^{(2)}. \tag{D.271}$$

Such a construction with two of the Pauli-matrices

$$\sigma^{(0)} = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \quad \text{and} \quad \sigma^{(2)} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \tag{D.272}$$

for an infinitesimally small angle suggest that any finite rotation by an angle α should be composable from n rotations by α/n in the limit $n \to \infty$:

$$R^i{}_a(\alpha) = \lim_{n\to\infty}\left(\sigma^{(0)} + \frac{\alpha}{n}\sigma^{(2)}\right)^n = \exp\left(\alpha\sigma^{(2)}\right) \tag{D.273}$$

where the matrix-valued exponential function is explained in terms of its series,

$$\begin{aligned} R^i{}_a = \exp\left(\alpha\sigma^{(2)}\right) &= \sum_n \frac{\alpha^n}{n!}\left(\sigma^{(2)}\right)^n = \sigma^{(0)}\sum_n \frac{\alpha^{2n}}{(2n)!}(-1)^n + \sigma^{(2)}\sum_n \frac{\alpha^{2n+1}}{(2n+1)!}(-1)^n \\ &= \sigma^{(0)}\cos\alpha + \sigma^{(2)}\sin\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \end{aligned} \tag{D.274}$$

which is the reason why $\sigma^{(2)}$ is referred to as the generator of all rotations, or equivalently, as the basis of the rotations as a Lie-group.

The same line of reasoning applies to Lorentz-transforms: They form likewise a Lie-group, parameterised by the rapidity ψ,

$$\begin{aligned} \Lambda(\psi) = \exp(\psi\sigma^{(3)}) &= \sum_n \frac{\psi^n}{n!}\left(\sigma^{(3)}\right)^n = \sigma^{(0)}\sum_n \frac{\psi^{2n}}{(2n)!} + \sigma^{(3)}\sum_n \frac{\psi^{2n+1}}{(2n+1)!} \\ &= \sigma^{(0)}\cosh\psi + \sigma^{(3)}\sinh\psi = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \end{aligned} \tag{D.275}$$

where the Pauli-matrix $\sigma^{(3)}$,

$$\sigma^{(3)} = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix} \tag{D.276}$$

can now be identified as the generator of the Lorentz-transforms. Comparing to the rotations one notices that the powers of $\sigma^{(3)}$ do not show changes in sign, but alternate between $\sigma^{(3)}$ for odd and $\sigma^{(0)}$ for even powers of n.

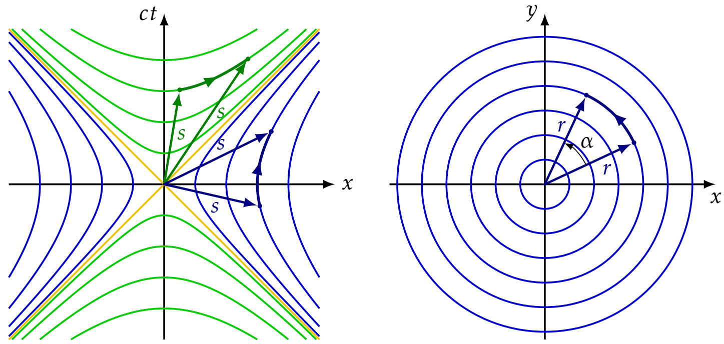


Figure 13: The rapidity ψ corresponds to the arc length that is covered by the end point of a vector with unit Minkowski norm $s^2 = (ct)^2 - x^2 = \pm 1$ under Lorentz transforms, in the same way as the rotation angle α is the arc length (or radian) covered by a point with unit Euclidean norm $r^2 = x^2 + y^2 = 1$ under a rotation. Reproduction with kind permission of I. Neutelings.

D.5 *Lorentz-group as a Lie-group*

It is intuitively clear that rotations form a group as subsequent rotations can be combined into a single rotations, and likewise, combinations of Lorentz transforms are Lorentz transforms again, Mathematically speaking, this is expressed by the group structure that is defined by the axioms: closedness, the existence of a unit element, the existence of an inverse element and lastly associativity.

For the closedness of a group one needs to show that the combination of group elements is again a group element. In a Lie-group, where the elements are generated by means of an exponential, one gets for instance for rotations

$$R(\alpha)R(\beta) = \exp\left(\alpha\sigma^{(2)}\right)\exp\left(\beta\sigma^{(2)}\right) = \left(\sum_i \frac{\alpha^i}{i!}\left(\sigma^{(2)}\right)^i\right)\left(\sum_j \frac{\beta^j}{j!}\left(\sigma^{(2)}\right)^j\right). \tag{D.277}$$

Multiplying the two exponential series can be achieved by application of the Cauchy-product

$$= \sum_i \sum_j^i \frac{\alpha^j}{j!}\frac{\beta^{i-j}}{(i-j)!}\left(\sigma^{(2)}\right)^j\left(\sigma^{(2)}\right)^{i-j} = \sum_i \frac{1}{i!}\left(\sum_j \binom{i}{j}\alpha^j\beta^{i-j}\right)\left(\sigma^{(2)}\right)^i \tag{D.278}$$

by using the definition of the binomial coefficient

$$\binom{i}{j} = \frac{i!}{j!(i-j)!}, \tag{D.279}$$

which leads to

$$= \sum_i \frac{(\alpha+\beta)^i}{i!}\left(\sigma^{(2)}\right)^i = R(\alpha+\beta) \tag{D.280}$$

by virtue of the generalised binomial formula,

$$(\alpha + \beta)^i = \sum_j \binom{i}{j} \alpha^j \beta^{i-j}, \tag{D.281}$$

which confirms the intuitive expectation that combining two rotations leads to a rotation again. In complete analogy one can show that $\Lambda(\psi)\Lambda(\varphi) = \Lambda(\psi + \varphi)$ for Lorentz transforms, with the rapidity as an additive parameter.

The unit element, which leaves a vector unchanged, is the quite obviously obtained for a rotation by the angle zero or a boost by zero rapidity:

$$R(\alpha = 0) = \exp(0 \times \sigma^{(2)}) = \exp(\sigma^{(2)})^0 = \mathrm{id} \tag{D.282}$$

Alternatively, one might argue that

$$R(\alpha = 0) = \sigma^{(0)} \cos(0) + \sigma^{(3)} \sin(0) = \sigma^{(0)} \equiv \mathrm{id} \tag{D.283}$$

and likewise obtain the unit matrix.

Associativity is very obvious for Lie-groups as their additive parameters naturally obey associativity:

$$R\left(\alpha + (\beta + \gamma)\right) = R\left((\alpha + \beta) + \gamma\right) \tag{D.284}$$

which implies

$$R\left(\alpha + (\beta + \gamma)\right) = R(\alpha)R(\beta + \gamma) = R(\alpha)\left[R(\beta)R(\gamma)\right] = \left[R(\alpha)R(\beta)\right]R(\gamma) = R(\alpha + \beta)R(\gamma) = R\left((\alpha + \beta) + \gamma\right) \tag{D.285}$$

Conservation of the norm of vectors under transformations, or equivalently, the orthogonality of the transform is realised in the following way, keeping in mind that $(\sigma^{(2)})^t = -\sigma^{(2)}$,

$$R^t(\alpha)R(\alpha) = \exp\left(\alpha\sigma^{(2)}\right)^t \exp\left(\alpha\sigma^{(2)}\right) = \exp\left(\alpha(\sigma^{(2)})^t\right)\exp\left(\alpha\sigma^{(2)}\right) = \exp\left(-\alpha\sigma^{(2)}\right)\exp\left(\alpha\sigma^{(2)}\right) = \exp\left((-\alpha + \alpha)\sigma^{(2)}\right) = \mathrm{id} \tag{D.286}$$

which differs slightly in the case of Lorentz-transforms, as they are orthogonal with respect to the the Minkowski-metric $\eta = \sigma^{(1)}$ instead of the Euclidean metric $\sigma^{(0)} = \mathrm{id}$,

$$\Lambda(\psi)^t \eta \Lambda(\psi) = \eta \quad \text{with} \quad \eta = \sigma^{(1)} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \tag{D.287}$$

Invariants of the transform such as determinants are realised in a funky way in Lie-groups: As an auxiliary result, we need that for any transform A with eigenvalues λ_i,

$$\ln \det(A) = \ln \prod_i \lambda_i = \sum_i \ln \lambda_i = \mathrm{tr} \ln(A) \tag{D.288}$$

where the matrix-valued logarithm ln(A) is naturally defined in terms of its series.

Because the logarithm can not be expanded at zero, where it is undefined, one uses this neat trick,

$$\ln(A) = \ln(\mathrm{id} + (A - \mathrm{id})) = \sum_n \frac{(-1)^{n+1}}{n}(A - \mathrm{id})^n. \tag{D.289}$$

Then,

$$\exp \ln \det(A) = \det(A) = \exp \operatorname{tr} \ln(A), \tag{D.290}$$

and with the substitution $B = \ln(A)$ one arrives at

$$\det \exp(B) = \exp \operatorname{tr}(B), \tag{D.291}$$

which is particular suitable for our purpose, as the determinant of a Lie-generated group element is related to the trace of its generator. Applied to the rotations this implies

$$\det(R) = \det \exp\left(\alpha\sigma^{(2)}\right) = \exp \operatorname{tr}\left(\alpha\sigma^{(2)}\right) = \exp\left(\alpha \operatorname{tr} \sigma^{(2)}\right) = \exp(0) = 1 \tag{D.292}$$

because the Pauli-matrix $\sigma^{(2)}$ is traceless. The same result for the Lorentz-transforms $\Lambda(\psi)$ follows in complete analogy,

$$\det(\Lambda) = \det \exp\left(\psi\sigma^{(3)}\right) = \exp \operatorname{tr}\left(\psi\sigma^{(3)}\right) = \exp\left(\psi \operatorname{tr} \sigma^{(3)}\right) = \exp(0) = 1. \tag{D.293}$$

Essentially, the determinant of the Lie-group is fixed to unity by the tracelessness of the generator.

Up to this point, we have been dealing with a single generator, but in 3+1 dimensions there might be cases where one combines rotations about different axes, boosts in different directions or even considers combinations between boosts and rotations! In these cases commutativity plays an important role, as it provides a correction factor to the rule $\exp(A)\exp(B) = \exp(A + B)$ known as the Baker-Hausdorff-Campbell formula:

$$\exp(A)\exp(B) = \exp(A + B)\exp\left(-\frac{1}{2}[A, B]\right), \tag{D.294}$$

with the commutator $[A, B] = AB - BA$.

D.6 *Adding velocities*

Subsequent Lorentz-transforms can be combined into a single transformation, and we already know that the Lorentz-transforms form a Lie-group with the rapidity ψ as an additive parameter instead of the velocity $\beta = \tanh\psi$. Luckily, there is a handy addition theorem for the hyperbolic tangent function:

$$\tanh(\psi + \varphi) = \frac{\tanh\psi + \tanh\varphi}{1 + \tanh\psi \cdot \tanh\varphi} \tag{D.295}$$

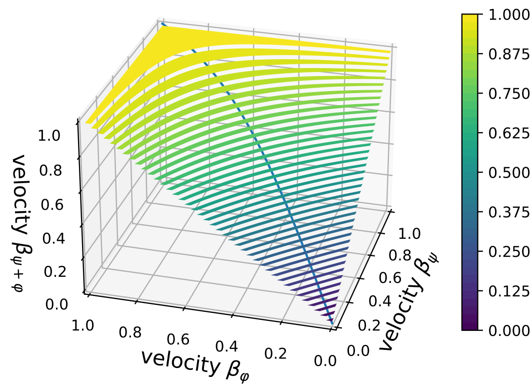


Figure 14: Relativistic addition theorem for velocities, with the particular case of $\beta_\psi = \beta_\varphi$ along the diagonal. $\beta_{\psi+\varphi}$ remains strictly below unity, excluding superluminal velocities. Clearly the relation needs to be linear if one of the velocities is zero, as seen at the edges.

Therefore, one obtains for the velocities

$$\beta_{\psi+\varphi} = \frac{\beta_\psi + \beta_\varphi}{1 + \beta_\psi \cdot \beta_\varphi} < 1 \tag{D.296}$$

leading to a combined velocity strictly smaller than the speed of light. Linearising the relationship shows a straightforward addition of velocities,

$$\beta_{\psi+\varphi} \simeq \beta_\psi + \beta_\varphi, \tag{D.297}$$

as one would expect from Galilean physics. A proof that the added velocities are strictly smaller than c might be done along these lines: Writing $\beta_\psi = 1 - x$ and $\beta_\varphi = 1 - y$ with positive x and y lead to

$$\beta_{\psi+\varphi} = \frac{(1-x)+(1-y)}{1+(1-x)(1-y)} = \frac{2-x-y}{2-x-y+xy} < 1 \tag{D.298}$$

because the product xy is larger than zero.

D.7 *Relativistic effects*

There are quite a number of relativistic effects, and they all hinge on the fact that spatial and temporal coordinates change jointly under Lorentz transforms, while only invariants constructed from them are truly fixed. If one chooses to ignore that the coordinates transform jointly and only looks at a single coordinate, surprising things will happen. Invariants will have identical values in all frames and take into account all coordinates. As such, they are the means for making statements that do

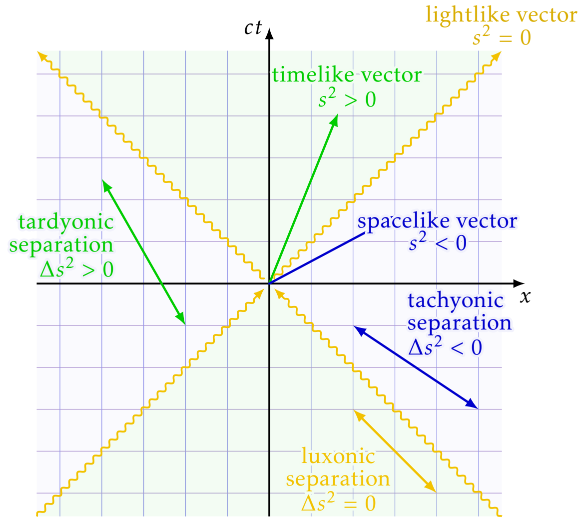


Figure 15: Classification of distances as spacelike $\Delta s^2 < 0$, timelike $\Delta s^2 > 0$ and lightlike $\Delta s^2 = 0$. Reproduction with kind permission of I. Neutelings.

not depend on a particular coordinate choice and hence transcend frames. Personally, I like skewed spacetime diagrams where the rapidities are chosen to be $\pm\psi/2$ because then the relative lengths in both frames are equal, and one can compare distances directly.

D.7.1 Constancy of the speed of light

In every frame, the speed of light comes out as constant, to the same numerical value, as illustrated by Fig. 16. This is no surprise, was it was the defining choice that differentiated Lorentz- from Galilei-transforms. In the diagram one immediately sees that a point on the diagonal, which corresponds to the light cone, acquires x- and ct-coordinates that change in proportionality to each other, indicating that their ratio is constant – the speed of light.

D.7.2 Relativity of simultaneity

Events at nonzero spatial separation, which take place at the same time (but at different positions), i.e. simultaneously on one frame, take place at different times in another frame, as shown in Fig. 17.

D.7.3 Time dilation

A time interval ct' taken at constant spatial coordinate x' gets mapped onto a time interval ct with differing spatial coordinates. The ratio between the two time intervals is proportional to the Lorentz-factor $\gamma \geq 1$. Fig. 18 shows how the time interval appears longer in projection.

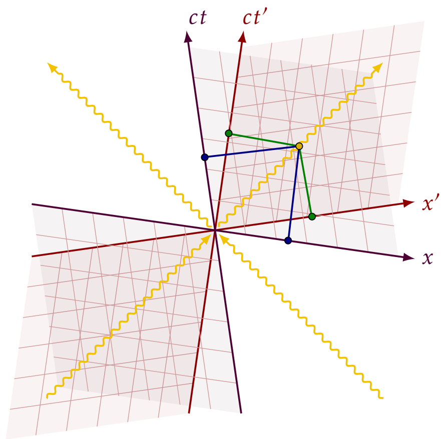


Figure 16: Constancy of the speed of light: The ratio of the two coordinates of a light-like event is always constant.

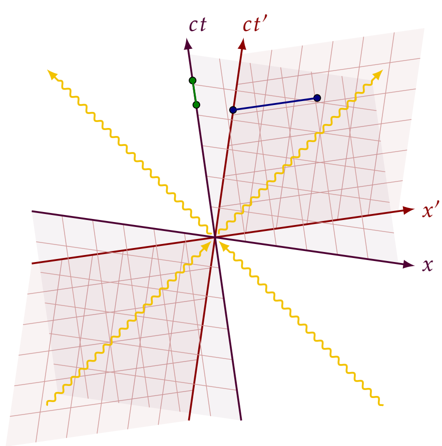


Figure 17: Relativity of simultaneity: Events that take place at the same time ct′ in one system (blue), take place at different times in another system (green).

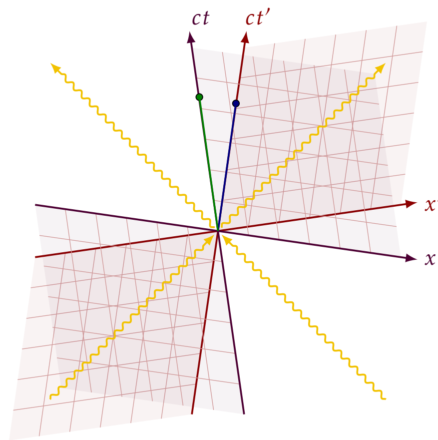


Figure 18: A duration ct′ of a process in one system (blue), seems to take more time when viewed from another system (green)

D.7.4 Length contraction

An object with a given length on one frame will appear to have a shorter length as viewed from another frame. This is ultimately traced back to the relativity of simultaneity: The length of an object is defined as the distance between its ends at the same time, but in a different frame, one effectively combines coordinates at different times, as demonstrated in Fig. 19. The contraction effect is proportional to the inverse Lorentz-factor $1/\gamma \leq 1$.

D.7.5 Causal ordering inside the light cone

The temporal order of time-like events is conserved under Lorentz-transforms, lightlike-events take place simultaneously, while the order of space-like separated events depends on the frame. To formulate this in a more extreme way, there is causal ordering only inside the light cone, and no causal ordering outside the light cone, as shown in Fig. 20.

D.8 *Proper time*

If a particle moves through spacetime along a trajectory $x^\mu(\tau)$ in the sense that it passes by the coordinates x^μ as its proper time τ evolves, one can define the 4-velocity u^μ of the particle as a tangent to the trajectory

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{D.299}$$

which is consistent with the definition of infinitesimal arc length $\mathrm{d}s$ along the trajectory, as

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \eta_{\mu\nu}u^\mu u^\nu\mathrm{d}\tau^2 = c^2\mathrm{d}\tau^2 \tag{D.300}$$

i.e. if the 4-velocity is defined with proper time τ as an affine parameter, it is normalised to $\eta_{\mu\nu}u^\mu u^\nu = c^2$, and the arc length is measurable by means of a clock.

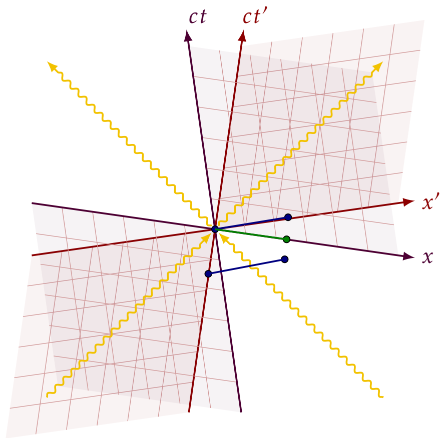


Figure 19: A yardstick at rest in the primed system (blue) seems to be contracted as viewed from another system (green).

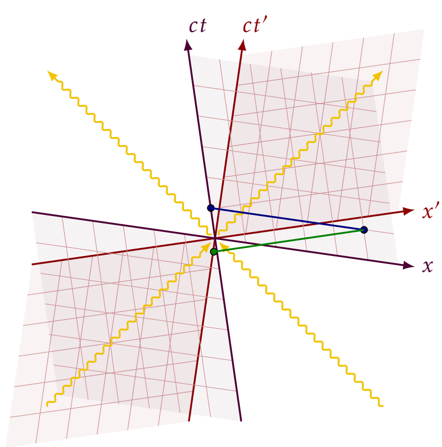


Figure 20: Spacelike separated events do not have an absolute causal ordering. The event seems to have a positive time coordinate ct (blue) and takes place after *the event at the origin, but a negative coordinate ct' (green) in the other frame and precedes the event at the origin.*

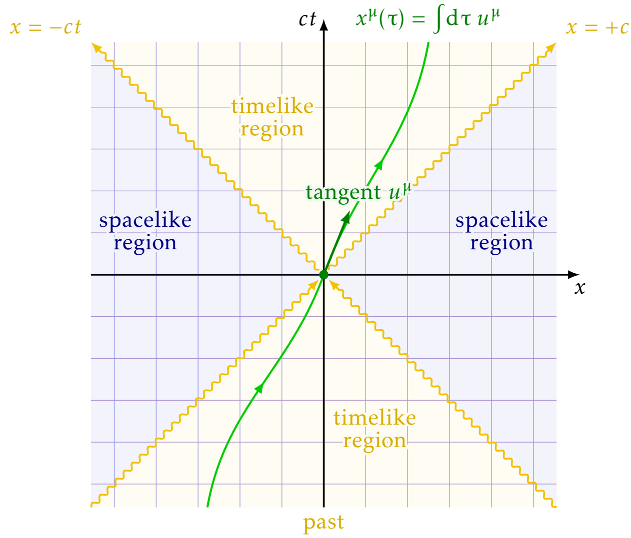


Figure 21: Spacetime diagram with spacelike (both left and right) and timelike (both past and future) regions, along with the worldline $x^\mu(\tau)$ of a massive particle, with 4-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. As $\eta_{\mu\nu}u^\mu u^\nu = c^2 > 0$, the massive particle necessarily moves inside the light cone. Reproduction with kind permission of I. Neutelings.

Proper time is the time elapsing on a clock that is carried along with the particle: The infinitesimal arc length can be expressed in terms of the coordinate differentials

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = c^2\mathrm{d}t^2 - \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = c^2\mathrm{d}\tau^2 \tag{D.301}$$

as the change of spatial coordinates $\mathrm{d}x^i$ is zero for the comoving clock. This implies three things: Proper time measures the arc length of the trajectory of a particle through spacetime,

$$s = \int_A^B \mathrm{d}s = c\int_A^B \mathrm{d}\tau = \int_A^B \sqrt{\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} \tag{D.302}$$

and is, as a Lorentz scalar, invariant under Lorentz transforms. And in addition, the normalisation of the 4-velocity is c^2 if τ is used as the affine parameter for $x^\mu(\tau)$.

Returning to the expression of s in terms of the infinitesimal coordinate changes leads to

$$s = \int_A^B \mathrm{d}s = c\int_A^B \mathrm{d}t\,\sqrt{1 - \frac{\gamma_{ij}}{c^2}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}} = c\int_A^B \mathrm{d}t\,\sqrt{1-\gamma_{ij}\beta^i\beta^j} = c\int_A^B \mathrm{d}t\,\frac{1}{\gamma} = c\int_A^B \mathrm{d}t\,\mathcal{L} \tag{D.303}$$

which can be used to compute arc lengths through spacetime.

Trajectories that have extremal values for s would result from a variational principle applied to $\mathcal{L} = 1/\gamma$. Hamilton's principle $\delta S = 0$ implies

$$\delta \int_A^B \mathrm{d}t\, \mathcal{L}(x^i, \upsilon^i) = \int_A^B \mathrm{d}t \left(\frac{\partial \mathcal{L}}{\partial x^i}\delta x^i + \frac{\partial \mathcal{L}}{\partial \upsilon^i}\delta \upsilon^i\right) = 0 \tag{D.304}$$

with the typical replacement

$$\delta \upsilon^i = \delta \frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\delta x^i \tag{D.305}$$

which enables integration by parts, yielding the Euler-Lagrange equation

$$\delta S = \int_A^B \mathrm{d}t \left(\frac{\partial \mathcal{L}}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \upsilon^i}\right) = 0 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \upsilon^i} = \frac{\partial \mathcal{L}}{\partial x^i} \tag{D.306}$$

The identical calculation can be done if the velocities are 4-velocities, expressed in terms of proper time τ

$$\delta \int_A^B \mathrm{d}\tau\, \mathcal{L}(x^\mu, u^\mu) = \int_A^B \mathrm{d}\tau \left(\frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial \mathcal{L}}{\partial u^\mu}\delta u^\mu\right) = 0 \tag{D.307}$$

with the typical replacement

$$\delta u^\mu = \delta \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\delta x^\mu \tag{D.308}$$

which enables integration by parts, yielding the Euler-Lagrange equation

$$\delta s = \int_A^B \mathrm{d}\tau \left(\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \mathcal{L}}{\partial u^\mu}\right) = 0 \quad \rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial \mathcal{L}}{\partial u^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{D.309}$$

for motion through 4-dimensional spacetime.

D.9 *Relativistic motion*

It would be a good idea to see if relativistic motion with the correct transformation property of all quantities involved would result naturally from a variational principle: This will be the case, and sometimes it appears to me that variational principles, always presented as the pinnacle of classical physics, are in fact relativistic: In some sense they are a piece of mathematics that has been discovered a few hundred years too early to appreciate them properly. They incorporate the idea that an invariant (under coordinate transforms) Lagrange-function gives rise to a covariant equation of motion. To see how this works, let's start at a classical Lagrange-function $\mathcal{L}(x^i, \upsilon^i)$

$$\mathcal{L}(x^i, \upsilon^i) = \frac{\gamma_{ij}}{2}\upsilon^i \upsilon^j - \Phi(x^i) \tag{D.310}$$

where both terms are invariant under e.g. rotations, Φ is scalar anyways and $\gamma_{ij}\upsilon^i\upsilon^j$ as the norm of the vector $\dot{x}$. Hamilton's principle $\delta S = 0$ with the action

$$S = \int_{t_i}^{t_f} \mathrm{d}t\, \mathcal{L}(x^i, \upsilon^i) \tag{D.311}$$

yields the Newtonian equation of motion

$$\ddot{x}^i = -\gamma^{ij}\partial_j\Phi \tag{D.312}$$

which sets the vector $\ddot{x}$ in relation with the gradient $\partial\Phi$, which is likewise a vector. While this is perfectly nice, there are some points of criticism for the variational principle that one can not answer from a classical point of view: There is no obvious interpretation of $\mathcal{L}$ or S, they are not measurable in a direct way and they behave funnily under Galilei transforms:

$$x^i \to x^i + u^i t \quad \text{and consequently} \quad \upsilon^i \to \upsilon^i + u^i \quad \text{for a constant relative velocity } u^i \tag{D.313}$$

This implies for the Lagrange function

$$\mathcal{L}(x^i, \upsilon^i) \to \frac{\gamma_{ij}}{2}\upsilon^i\upsilon^j + \gamma_{ij}\upsilon^i u^j + \frac{\gamma_{ij}}{2}u^i u^j = \frac{\gamma_{ij}}{2}\upsilon^i\upsilon^j + \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_{ij}x^i u^j + \frac{\gamma_{ij}}{2}u^i u^j t\right) \tag{D.314}$$

In fact, the Lagrange function is not invariant under Galilei-transforms, but the additional term appearing is a total time derivative and does therefore not play a role in the variational principle. It might strike you as odd (and rightfully so), that rotations and Galilei-transforms are treated so differently.

Thinking about a relativistic Lagrange-function that should be intuitive, measurable and invariant leads to proper time

$$c\tau = c\int_A^B \mathrm{d}\tau = \int_A^B \mathrm{d}s = \int_A^B \sqrt{\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} \tag{D.315}$$

It is the time that is displayed as elapsed on a clock that is moving along with the particle and is, geometrically, the arc length of the trajectory through spacetime, measured with the Minkowski-metric $\eta_{\mu\nu}$. As this metric defines an invariant, the arc-length $c\tau = s$ will be identical in any Lorentz frame, and it will be a convex functional in the velocity $\upsilon = c\beta$, making sure that the variational principle finds a uniquely defined minimum and enabling Legendre-transforms to find the associated energy. As affine transformations $\mathcal{L} \to a\mathcal{L} + b$ of the Lagrange-function or the action do not have any influence on the Euler-Lagrange-equation, we can include a prefactor $-mc$ to yield

$$S = -mc^2\int_A^B \mathrm{d}\tau = -mc\int_A^B \mathrm{d}s = -mc\int_A^B \frac{\mathrm{d}t}{\gamma} \quad \to \quad \mathcal{L} = -\frac{mc}{\gamma} \tag{D.316}$$

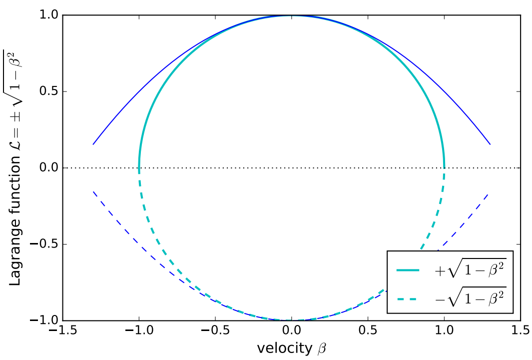


Figure 22: Relativistic Lagrange function $\mathcal{L}(\beta) = \pm\sqrt{1-\beta^2}$ in comparison to its classical limits $\mathcal{L}(\beta) = \pm 1 \mp \beta^2/2$.

where the difference between the arc length s and the action S has vanished, or in other words: We've found a geometric interpretation of the action.

It is very instructive to reformulate time proper time integral in terms of the 4-velocity u^μ,

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \gamma \begin{pmatrix} c \\ \upsilon^i \end{pmatrix} \quad \text{for} \quad x^\mu = \begin{pmatrix} ct \\ x^i \end{pmatrix} \tag{D.317}$$

with the definition of the conventional velocity as $\upsilon^i = \mathrm{d}x^i/\mathrm{d}t$. Then,

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\mathrm{d}\tau^2 = \gamma^2\left(c^2 - \upsilon_i\upsilon^i\right)\mathrm{d}\tau^2 = c^2\underbrace{\gamma^2(1-\beta^2)}_{=1}\mathrm{d}\tau^2 = c^2\mathrm{d}\tau^2 \tag{D.318}$$

and the normalisation of the 4-velocity is timelike, $\eta_{\mu\nu}u^\mu u^\nu = c^2 > 0$, as the particle moves necessarily inside the light cone.

D.10 *Relativistic dispersion relations*

With the relativistic Lagrange function $\mathcal{L}$ being equal to the inverse Lorentz-factor,

$$\mathcal{L} = -\frac{1}{\gamma} = -\sqrt{c^2 - \upsilon^2} \tag{D.319}$$

one can derive the canonical momentum p

$$p = \frac{\partial\mathcal{L}}{\partial\upsilon} = \frac{\upsilon}{\sqrt{c^2-\upsilon^2}} \quad \text{such that} \quad \upsilon = \frac{cp}{\sqrt{1+p^2}} \tag{D.320}$$

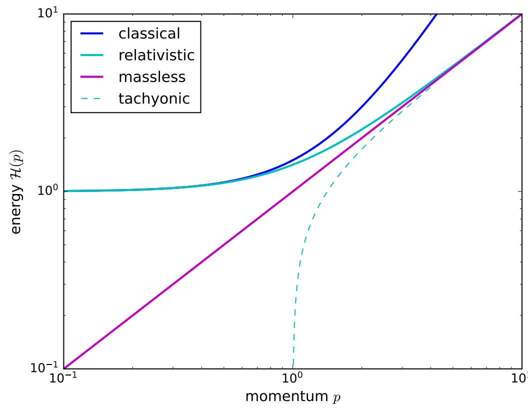


Figure 23: Relativistic dispersion relation $\mathcal{H} = \sqrt{p^2 + m^2}$ for $m = 1$, in comparison to the massless dispersion relation $\mathcal{H} = p$, the classical dispersion relation $\mathcal{H} = p^2/2 + 1$ and a tachyonic dispersion relation $\mathcal{H} = \sqrt{p^2 - 1}$, which is only defined for $p \geq 1$.

Carrying out the Legendre-transform for obtaining $\mathcal{H}$ from $\mathcal{L}$

$$\mathcal{H} = \upsilon(p)p - \mathcal{L}(\upsilon(p)) \tag{D.321}$$

then implies

$$\mathcal{H} = \upsilon\frac{\upsilon}{\sqrt{c^2 - \upsilon^2}} + \sqrt{c^2 - \upsilon^2} = \upsilon p + \frac{\upsilon}{p} = \upsilon\left(p + \frac{1}{p}\right) = c\frac{p}{\sqrt{1 + p^2}}\frac{1 + p^2}{p} = c\sqrt{1 + p^2} \tag{D.322}$$

and if one would include mc as a prefactor,

$$\mathcal{H} = \sqrt{(cp)^2 + (mc^2)^2} \tag{D.323}$$

which is exactly the relativistic dispersion relation. Surprisingly, the energy $\mathcal{H}$ is not zero even for $p = 0$, which is why we associate this energy mc^2 to the rest mass of a particle. With this dispersion relation it is straightforward to compute the group and phase velocities of a wave packet associated with a relativistic particle,

$$\upsilon_{\mathrm{gr}} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}p} = c^2\frac{p}{\mathcal{H}} \quad \text{and} \quad \upsilon_{\mathrm{ph}} = \frac{\mathcal{H}}{p} \tag{D.324}$$

such that their geometric average is exactly c^2:

$$\upsilon_{\mathrm{gr}} \times \upsilon_{\mathrm{ph}} = c^2 \tag{D.325}$$

Because for any momentum $\mathcal{H} > cp$, it is the case that $\upsilon_{\mathrm{gr}} < c$ while $\upsilon_{\mathrm{ph}} > c$. It is reassuring to see that the group velocity, associated with the motion of massive particles, is always subluminal.

The Euler-Lagrange equation for minimising the arc-length $s = \int \mathrm{d}s$ reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial u^\alpha} = \frac{\partial\mathcal{L}}{\partial x^\alpha} \quad \text{for} \quad \mathcal{L} = \sqrt{\eta_{\mu\nu}u^\mu u^\nu} \tag{D.326}$$

where the right side is automatically zero in this case, because $\mathcal{L}$ does not depend on x^α. Evaluating the left side gives:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2c}\eta_{\mu\nu}\left[\frac{\partial u^\mu}{\partial u^\alpha}u^\nu + u^\mu\frac{\partial u^\nu}{\partial u^\alpha}\right]\right) = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2c}\eta_{\mu\nu}\left[\delta^\mu_\alpha u^\nu + u^\mu\delta^\nu_\alpha\right]\right) = \frac{1}{2c}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\eta_{\alpha\nu}u^\nu + \eta_{\mu\alpha}u^\mu\right) = \frac{1}{c}\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} = 0 \tag{D.327}$$

implying that in the absence of forces, the particle moves through spacetime at a constant 4-velocity, or equivalently, that a straight line corresponds to motion free of acceleration: This is exactly the relativistic version of Newton's law of inertia. And it remains true, even in Minkowski-space, that inertial motion along a straight line minimises the arc length: The straightest trajectory is the shortest. It is quite astonishing to see the geometric picture behind Newton's axioms that is somewhat hidden in classical mechanics.

Expanding the arc length s in terms of a Taylor-expansion for small velocities

$$s = \int_A^B \mathrm{d}s = c\int_A^B \mathrm{d}t\,\sqrt{\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}} = c\int_A^B \mathrm{d}t\,\sqrt{c^2 - \upsilon^2} \simeq \int_A^B \mathrm{d}t\left(1 - \frac{\upsilon^2}{2}\right) \tag{D.328}$$

recovering the square of the velocity familiar from classical mechanics, in the approximation $\sqrt{1-\beta^2} \simeq 1 - \beta^2/2$ for $\beta \ll 1$. Weirdly enough, we see that it doesn't have anything to do with kinetic energies, it is just the lowest-order Taylor-expansion of the relativistic arc length and is a purely geometrical object. With the suggestive identification of the arc length as the action and the line element or proper time interval as the Lagrange function, one really falls back onto the kinetic energy as the Lagrange function of classical mechanics, because it is only ever defined up to an affine transform, negating the influence of the additive 1, and allowing to multiply the line element with the negative mass.

E COVARIANT ELECTRODYNAMICS

E.1 *Covariant formulation of electrodynamics*

Relativity provides the tools to formulate the Maxwell-equations very compactly, elegantly, and in a Lorentz-covariant way. For this purpose, one needs to construct a differential operator ∂_μ for derivatives with respect to the coordinates, which themselves form a Lorentz-vector x^μ.

⚠ Sometimes, ∂^μ is used, defined as $\partial^\mu = \eta^{\mu\nu}\partial_\nu$, but please avoid notations like $\partial^\mu = \partial/\partial x_\mu$.

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_{ct}, +\partial_i) \tag{E.329}$$

For consistency, the divergence $\partial_\mu x^\mu$ needs to be equal to the dimensionality

$$\partial_\mu x^\mu = \frac{\partial x^\mu}{\partial x^\mu} = \partial_{ct}(ct) + \partial_i x^i = 4 \tag{E.330}$$

which comes out naturally. With this differential form ∂_μ, the d'Alembert-operator is given as a Lorentz-square,

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_{ct}^2 - \gamma^{ij}\partial_i\partial_j = \partial_{ct}^2 - \Delta, \tag{E.331}$$

and is in fact a Lorentz-scalar, as shown by the orthogonality relation of the Lorentz-transforms,

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu \rightarrow \underbrace{\eta^{\mu\nu}\Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta}_{=\eta^{\alpha\beta}}\partial_\alpha\partial_\beta = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = \Box, \tag{E.332}$$

reflecting the fact that wave propagation according to $\Box$ takes place at the velocity c in every frame, which was the defining principle of the Lorentz transforms. The transformation property $\partial_\mu \rightarrow \Lambda_\mu{}^\alpha\partial_\alpha$ generalises the transformation $\partial_i \rightarrow \mathrm{R}_i{}^j\partial_j$ to the full Lorentz group. In the same way as Δ is invariant under rotations, $\Box$ becomes invariant under combined rotations and Lorentz transforms.

With the operator ∂_μ it is straightforward to formulate the continuity equation for the charge density:

⚠ $\jmath^\mu$ contains the electric charge density ρ and the current density $\jmath^i$ as a vector.

$$\jmath^\mu = \begin{pmatrix} \rho c \\ \jmath^i \end{pmatrix} \quad \text{with} \quad \partial_\mu \jmath^\mu = \partial_{ct}(\rho c) + \partial_i \jmath^i = 0 \tag{E.333}$$

where it is interesting to see, that $\jmath^t = \rho c$ has the same units as $\jmath^i$, reflecting the consistency of the units in ∂_{ct} and ∂_i, with the additional benefit that a charge at rest in a given frame has a nonzero t-component $\jmath^t = c\rho$, as it moves with the velocity c along the ct-axis!

As a Lorentz-vector, the 4-current density transforms according to

$$\jmath^\mu \rightarrow \Lambda^\mu{}_\alpha \jmath^\alpha \tag{E.334}$$

and necessarily inversely to ∂_μ, such that $\partial_\mu \jmath^\mu$ is indeed a Lorentz-scalar and has the same value in all Lorentz-frames: The derivative transforms according to $\partial_\mu \rightarrow \Lambda_\mu{}^\alpha\partial_\alpha$ and the vectorial $\jmath^\mu$ inversely, $\jmath^\mu \rightarrow \Lambda^\mu{}_\alpha \jmath^\alpha$, such that

$$\partial_\mu \jmath^\mu \to \Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta \partial_\alpha \jmath^\beta = \delta^\alpha_\beta \partial_\alpha \jmath^\beta = \partial_\alpha \jmath^\alpha \tag{E.335}$$

with $\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = \delta^\alpha_\beta$, as the two Lorentz-transforms are inverse to each other.

This differential formulation with its clear Lorentz-invariance has a giant advantage over an integral formulation within a given frame: Earlier, we would have written

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V\,\rho = -\int_{\partial V} \mathrm{d}S_i\,\jmath^i. \tag{E.336}$$

Observed from a different Lorentz frame, the integration volume V is relativistically contracted by a Lorentz-factor γ, while the charge density ρ is larger by the same factor, as the charge is squeezed into a seemingly smaller volume. The two effects compensate each other, after all, it is the same charge within V. The surface ∂V of the volume is smaller by γ, too, for this to be true one can easily imagine a cuboid which is contracted by γ along the direction of motion. But for the same reasons as for the charge density, the current density $\jmath$ is changed by the inverse factor. Lastly, there is relativistic time dilation appearing in $\mathrm{d}/\mathrm{d}t$ as well as in the current density $\jmath^i$, again compensating each other: One sees all charge carries changing position at a slower rate due to their dilated proper time, leading to smaller fluxes $\jmath^i$ and smaller rates of change of ρ.

E.2 *Maxwell's equations*

E.2.1 Inhomogeneous Maxwell equations

The inhomogeneous Maxwell-equations are first of all a divergence $\partial_i D^i = 4\pi\rho$ and a rotation $\epsilon^{ijk}\partial_j H_k = +\partial_{ct} D^i + 4\pi/c\,\jmath^i$. But with the help of the dual tensor $H^{ij} = \epsilon^{ijk} H_k$ the first term of Ampère's law becomes a divergence as well, $\epsilon^{ijk}\partial_j H_k = \partial_j H^{ij}$. This motivates to package the two equations into a single divergence-like tensorial relation,

$$\partial_\mu G^{\mu\nu} = \frac{4\pi}{c}\jmath^\nu, \quad \text{in components} \quad G^{\mu\nu} = \begin{pmatrix} 0 & +D^x & +D^y & +D^z \\ -D^x & 0 & +H^z & -H^y \\ -D^y & -H^z & 0 & +H^x \\ -D^z & +H^y & -H^x & 0 \end{pmatrix} \tag{E.337}$$

with the antisymmetric field tensor $G^{\mu\nu}$. When inspecting the coordinates separately, one obtains $\partial_\mu G^{\mu t} = \partial_i D^i = 4\pi/c\,\jmath^t = 4\pi\rho$ and $\partial_\mu G^{\mu i} = -\partial_{ct} D^i + \epsilon^{ijk}\partial_j H_k = 4\pi/c\,\jmath^i$.

⚠ *$G^{\mu\nu}$ contains the fields D^i and H_i (effectively as $H^{ij} = \epsilon^{ijk}H_k$) in matter.*

One of the first conclusion we drew from the Maxwell-equations was that the field respected charge conservation, which becomes very apparent in this formalism:

$$\partial_\mu G^{\mu\nu} = \frac{4\pi}{c}\jmath^\nu \to \partial_\nu\partial_\mu G^{\mu\nu} = \frac{4\pi}{c}\partial_\nu \jmath^\nu = 0 \tag{E.338}$$

implying that the continuity equation $\partial_\nu \jmath^\nu = 0$ is valid because of the contraction of the symmetric operator $\partial_\nu\partial_\mu$ with an antisymmetric tensor $G^{\mu\nu}$. With 6 free entries as an antisymmetric tensor, $G^{\mu\nu}$ can accommodate 3 components of the electric field D^i and 3 components of the magnetic field H_i.

⚠ *It follows from the antisymmetry of $G^{\mu\nu}$ that in $n+1$ dimensions, there would be n components for D^i but $n(n-1)/2$ components for H_i.*

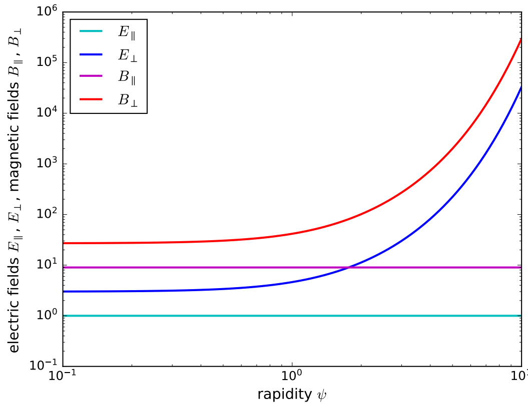


Figure 24: Electric and magnetic field components under Lorentz boosting $\tilde{F}^{\alpha\beta} \rightarrow \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\tilde{F}^{\mu\nu}$ *as a function of rapidity* ψ.

E.2.2 Homogeneous Maxwell equations

Writing the two homogeneous Maxwell-equations as divergences requires a similar construction: For that purpose, one defines the dual field tensor $\tilde{F}^{\mu\nu}$ with a suitable arrangement of the fields E_i and B^i: The rotation appearing in the induction law is recast into a divergence $\epsilon^{ijk}\partial_j E_k = \partial_j \epsilon^{ijk} E_k = \partial_j E^{ij}$ with the dual $E^{ij} = \epsilon^{ijk}E_k$. Combining the electric field components in a similar alternating fashion with the magnetic field components leads to,

⚠ $\tilde{F}^{\mu\nu}$ *contains the fields* B^i *and* E_i *(effectively as* $E^{ij} = \epsilon^{ijk}E_k$*) in vacuum.*

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \text{in components} \quad \tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ +B_x & 0 & +E_z & -E_y \\ +B_y & -E_z & 0 & +E_x \\ +B_z & +E_y & -E_x & 0 \end{pmatrix}. \tag{E.339}$$

With this definition of the dual field tensor, one can write analogously $\partial_\mu \tilde{F}^{\mu t} = \partial_i B^i = 0$ (the overall minus-sign does not matter) and $\partial_\mu \tilde{F}^{\mu i} = \partial_{ct} B^i + \epsilon^{ijk}\partial_j E_k = 0$. Electromagnetic duality in vacuum now amounts simply to interchanging $G^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$, because $\partial_\mu G^{\mu\nu} = \partial_\mu \tilde{F}^{\mu\nu} = 0$ as soon as $\jmath^\nu = 0$.

⚠ *Again, antisymmetry of* $\tilde{F}^{\mu\nu}$ *requires that in* $n+1$ *dimensions, there would be* n *components for* B^i *but* $n(n-1)/2$ *components for* E_i.

Both field tensors transform under boosting according to $\tilde{F}^{\mu\nu} \rightarrow \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\tilde{F}^{\alpha\beta}$ and $G^{\mu\nu} \rightarrow \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}G^{\alpha\beta}$, which has a strong effect $\propto \gamma^2$ (in fact, two Λs needed because the tensors have two indices) on the fields, as illustrated in Fig. 24.

E.3 *Relativistic potentials and gauging*

The next step would be to package the potentials Φ and A_i into a 4-potential, according to

$$A_\mu = (\Phi, -A_i)\,, \tag{E.340}$$

which allows to write the Lorenz gauge-condition in a very compact way as a divergence:

$$\eta^{\mu\nu}\partial_\mu A_\nu = \partial_{ct}\Phi + \gamma^{ij}\partial_i A_j = 0, \tag{E.341}$$

A_μ contains the electric potential Φ and the magnetic potential A_i as a linear form.

where the minus signs from the spatial part of the metric $\eta^{\mu\nu}$ and of the spatial part of A_μ cancel each other. Defining the potential A_μ as in eqn. (E.340) allows to write wave equation in Lorenz-gauge in a very compact form,

$$\Box A_\mu = \frac{4\pi}{c}\eta_{\mu\nu}\jmath^\nu, \tag{E.342}$$

which at the same time explains the minus-sign in the spatial part of A_μ as well as the cancellation of the additional factor of c in $\jmath^t = \rho c$.

Linking the potential A_μ to the Faraday tensor $F_{\mu\nu}$ is possible by writing

$$\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}, \tag{E.343}$$

because then the the electric field components would be given as $F_{it} = \partial_i A_t - \partial_{ct} A_i = -\partial_i \Phi - \partial_{ct} A_i = E_i$ as well as $F_{ij} = \partial_i A_j - \partial_j A_i$ with mutually different indices (ijk). It is interesting to see, how the requirement of antisymmetry reduces the number of free field components from initially 16 in $\partial_\mu A_\nu$ to 6, corresponding to 3 components of the electric and 3 components of the magnetic field. Weirdly enough, it's a bit of a coincidence that in 3 + 1 dimensions there are as many components of the electric and of the magnetic field, allowing to write B^i as a vector:

$$B^i = \epsilon^{ijk}F_{jk} = \epsilon^{ijk}\left(\partial_j A_k - \partial_k A_j\right) \tag{E.344}$$

albeit with a small caveat: Under parity transform $\mathcal{P}$, B^i does not change its sign, because both ∂_i and A_j change their signs. In contrast, E_i does change its sign, because in $\partial_i\Phi$ only ∂_i changes its sign, and in $\partial_{ct}A_i$ only A_i! Consequently, one calls E_i a polar vector and B^i an axial vector.

Applying gauge transformations would change the potentials, $A_\mu \rightarrow A_\mu + \partial_\mu\chi$, but leaves the Faraday tensor $F_{\mu\nu}$ invariant, as

$$F_{\mu\nu} \rightarrow \partial_\mu\left(A_\nu + \partial_\nu\chi\right) - \partial_\nu\left(A_\mu + \partial_\mu\chi\right) = \partial_\mu A_\nu - \partial_\nu A_\mu + \underbrace{\partial_\mu\partial_\nu\chi - \partial_\nu\partial_\mu\chi}_{=0} = F_{\mu\nu} \tag{E.345}$$

as partial derivatives interchange. The same result applies to the tensor $G^{\mu\nu}$ as it originates from $F_{\mu\nu}$ through a linear transform. It is well possible to derive $\tilde{F}^{\mu\nu}$ from the potential directly, through

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) = \epsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta, \tag{E.346}$$

using an antisymmetry-argument in the second step. Gauge transforms on the potential imply

$$\epsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta \rightarrow \epsilon^{\mu\nu\alpha\beta}\partial_\alpha\left(A_\beta + \partial_\beta\chi\right) = \epsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta + \epsilon^{\mu\nu\alpha\beta}\partial_\alpha\partial_\beta\chi = \epsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta = \tilde{F}^{\mu\nu} \tag{E.347}$$

with the contraction of the symmetric $\partial_\alpha\partial_\beta$ with the antisymmetric $\epsilon^{\mu\nu\alpha\beta}$ vanishes. In consequence, not only $F_{\mu\nu}$ but also $\tilde{F}^{\mu\nu}$ is gauge-invariant, and by extension $\tilde{G}_{\mu\nu}$.

An interesting manipulation shows a derivative relation for $F_{\mu\nu}$ as it originates from the potential. Composing a cyclic permutation of indices in $\partial_\lambda F_{\mu\nu}$ yields

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = \partial_\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\mu(\partial_\nu A_\lambda - \partial_\lambda A_\nu) + \partial_\nu(\partial_\lambda A_\mu - \partial_\mu A_\lambda) = 0 \tag{E.348}$$

with a pairwise cancellation of the terms. This derivative relation is called the Bianchi-identity and is in fact equivalent to the field equation $\partial_\mu\tilde{F}^{\mu\nu} = 0$ for the dual tensor $\tilde{F}^{\mu\nu}$,

$$\partial_\mu\tilde{F}^{\mu\nu} = \frac{\partial_\mu}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \frac{\partial_\mu}{2}\epsilon^{\mu\nu\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = \partial_\mu\epsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta = \epsilon^{\mu\nu\alpha\beta}\partial_\mu\partial_\alpha A_\beta = 0, \tag{E.349}$$

with the well-used argument that a contraction between a symmetric and an anti-symmetric index pair, here (α,μ), has to vanish. One sees immediately, that working with a potential is enabled by the condition $\partial_\mu\tilde{F}^{\mu\nu} = 0$ instead of $\tilde{F}^{\mu\nu}$ being sourced by a magnetic charge density $\imath^\nu$, in the spirit of

$$\partial_\mu\tilde{F}^{\mu\nu} = -\frac{4\pi}{c}\imath^\nu, \tag{E.350}$$

with an associated conservation law $\partial_\nu\imath^\nu = 0$. Only then can we make the argument that a potential A_μ invalidates a nonzero divergence of $\tilde{F}^{\mu\nu}$.

The field tensor $G^{\mu\nu}$ containing D^i and H_i can be related to the field tensor $F_{\mu\nu}$ containing E_i and B^i by means of a generalised constitutive relation,

$$G^{\alpha\beta} = X^{\alpha\beta\mu\nu}F_{\mu\nu} \quad\leftrightarrow\quad F_{\alpha\beta} = X_{\alpha\beta\mu\nu}G^{\mu\nu} \tag{E.351}$$

with the orthogonality relation

$$X^{\alpha\beta\mu\nu}X_{\mu\nu\gamma\delta} = \delta^\alpha_\gamma\,\delta^\beta_\delta, \quad \text{implying} \quad G^{\alpha\beta} = X^{\alpha\beta\mu\nu}X_{\mu\nu\gamma\delta}\,G^{\gamma\delta} = \delta^\alpha_\gamma\delta^\beta_\delta\,G^{\gamma\delta} = G^{\alpha\beta} \tag{E.352}$$

The tensor $X^{\alpha\beta\mu\nu}$ is antisymmetric in each index pair (α,β), (μ,ν) and maps an antisymmetric linear form $F_{\mu\nu}$ to an antisymmetric vectorial tensor $G^{\mu\nu}$. Tensors of that type can be written as being proportional to proper antisymmetrisations of the metric,

$$X^{\alpha\beta\mu\nu} = \frac{\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}}{2}, \tag{E.353}$$

allowing us to convert the divergence $\partial_\mu G^{\mu\nu} = 4\pi/c\,\jmath^\nu$ into a wave equation for the potentials,

$$\partial_\mu G^{\mu\nu} = \partial_\mu X^{\mu\nu\alpha\beta} F_{\alpha\beta} = \partial_\mu X^{\mu\nu\alpha\beta}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) = 2\partial_\mu X^{\mu\nu\alpha\beta}\partial_\alpha A_\beta =$$
$$\left(\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}\right)\partial_\mu\partial_\alpha A_\beta = \underbrace{\eta^{\alpha\mu}\partial_\mu\partial_\alpha}_{=\Box}\eta^{\beta\nu}A_\beta - \eta^{\alpha\nu}\partial_\alpha\underbrace{\eta^{\beta\mu}\partial_\mu A_\beta}_{=0} = \Box\eta^{\beta\nu}A_\beta = \frac{4\pi}{c}\jmath^\nu. \tag{E.354}$$

In summary, under the assumption of Lorenz-gauge, the wave equation

$$\Box A_\beta = \frac{4\pi}{c}\eta_{\beta\nu}\jmath^\nu \tag{E.355}$$

relates potential and source, where we have already discussed solutions in terms of Liénard-Wichert retarded potentials. Effectively, with the time-component of the source being $c\rho$, and the overall coupling constant being $4\pi/c$, one can combine both potentials into a single linear form and all sources into a single vector.

E.4 *Dual field tensors and the Bianchi-identity*

The duality transformation interchanges the positions of the electric and magnetic field components when transitioning from $F_{\mu\nu}$ to $\tilde{F}^{\mu\nu}$ and vice versa:

$$\tilde{F}^{\alpha\beta} = -\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu} \quad\leftrightarrow\quad F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\tilde{F}^{\alpha\beta} \tag{E.356}$$

making $F_{\mu\nu}$ autodual

$$\tilde{\tilde{F}}_{\mu\nu} = -\frac{1}{4}\epsilon_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = \delta^{\gamma\delta}_{\mu\nu}F_{\gamma\delta} = \frac{1}{2}\left(\delta^\gamma_\mu\delta^\delta_\nu - \delta^\gamma_\nu\delta^\delta_\mu\right)F_{\gamma\delta} = \frac{1}{2}\left(F_{\mu\nu} - F_{\nu\mu}\right) = F_{\mu\nu}, \tag{E.357}$$

where analogous formulas apply to $\tilde{F}^{\mu\nu}$. For the contraction between the two Levi-Civita symbols we have used the relation

$$\epsilon^{i_1\ldots i_q k_1\ldots k_p}\epsilon_{k_1\ldots k_p j_1\ldots j_q} = -p!q!\delta^{i_1\ldots i_q}_{j_1\ldots j_q}, \tag{E.358}$$

valid for Minkowksi-spaces, with the dimension $n = p + q$ and the overlap p between the indices to be contracted. Specifically, we need $p = 2 = q$ in $n = 4$. $\delta^{i_1\ldots i_q}_{j_1\ldots j_q}$ refers to the generalised Kronecker symbol. In complete analogy, there is a dual $\tilde{G}_{\mu\nu}$ of the field tensor $G^{\mu\nu}$,

$$\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta} \quad\leftrightarrow\quad G^{\alpha\beta} = -\frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\tilde{G}_{\mu\nu}. \tag{E.359}$$

To make things more concrete, one can follow through how the duality transform reorganises the tensors $\tilde{F}^{\mu\nu}$ and $G^{\mu\nu}$ isolated from the homogeneous and inhomogeneous Maxwell-equations. First of all, $\epsilon_{\alpha\beta\mu\nu}\tilde{F}^{\mu\nu}$ maps the antisymmetric (μ,ν) index pair to an object $F_{\alpha\beta}$, which is likewise antisymmetric, this time in (α,β). For a non-vanishing contribution, all indices in the Levi-Civita-symbol need to be different, which implies that there is no linear combination being formed, but simply a remapping of all components: For instance, choosing $(\alpha,\beta) = (t,x)$ for $F_{\alpha\beta}$ can only acquire a combination from $\tilde{F}^{\mu\nu}$ for $(\mu,\nu) = (y,z)$ or (z,y). But $\tilde{F}^{y,z} = -\tilde{F}^{z,y}$ due to the antisymmetry of the field tensor, therefore the two are equal, and are added twice, which in turn is remedied by the prefactor of 1/2.

Specifically F_{tx} will be set equal to $\tilde{F}^{yz} = E_x$, and F_{xy} will become $\tilde{F}^{tz} = -B_z$: We observe, how the first row and the first column of $F_{\alpha\beta}$ will accommodate the electric field components which had been stored in the interior of the tensor $\tilde{F}^{\mu\nu}$, while the first row and first column of $\tilde{F}^{\mu\nu}$ get scattered into the interior of the tensor $F_{\alpha\beta}$: Effectively, the magnetic and electric field components get interchanged up to a sign, leading to:

$$F_{\mu\nu} = \begin{pmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & -B_z & +B_y \\ -E_y & +B_z & 0 & -B_x \\ -E_z & -B_y & +B_x & 0 \end{pmatrix}. \tag{E.360}$$

The same rearrangement takes place in the duality transform of the tensor $G^{\alpha\beta}$:

$$\tilde{G}_{\mu\nu} = \begin{pmatrix} 0 & -H^x & -H^y & -H^z \\ +H^x & 0 & -D^z & +D^y \\ +H^y & +D^z & 0 & -D^x \\ +H^z & -D^y & +D^x & 0 \end{pmatrix} \tag{E.361}$$

with the replacement of D^i and H_i, again with a sign change: This sign change is very important, as it recovers the idea of duality of electromagnetism in vacuum, where under the replacement of electric and magnetic fields the Maxwell equations do not change.

The duality transform respects the antisymmetry of $\tilde{F}^{\mu\nu}$ and $F_{\mu\nu}$, which is important because it links charge conservation to gauge invariance of the potentials: Nature has chosen to have $\imath^\mu = 0$ and $\partial_\mu \imath^\mu = 0$ which has important implications, as we can now differentiate between the inhomogeneous and homogeneous Maxwell equations, which read:

$$\partial_\mu G^{\mu\nu} = \frac{4\pi}{c} \jmath^\nu \quad \text{and} \quad \partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{E.362}$$

With $F_{\mu\nu}$ following from a potential A_μ in an antisymmetrised, gauge-invariant way,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{E.363}$$

the homogeneous Maxwell equation is automatically fulfilled, as

$$\partial_\mu \tilde{F}^{\mu\nu} = \frac{1}{2} \partial_\mu \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \epsilon^{\mu\nu\alpha\beta} \partial_\mu \partial_\alpha A_\beta = 0 \tag{E.364}$$

through the contraction of the antisymmetric Levi-Civita symbol over the symmetric index pair (α,μ).

The equivalence of the Bianchi-identity

$$\partial_\mu F_{\alpha\beta} + \partial_\beta F_{\mu\alpha} + \partial_\alpha F_{\beta\mu} = 0 \tag{E.365}$$

and the divergence-like field equation for the dual tensor $\tilde{F}^{\mu\nu}$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \tag{E.366}$$

can be shown as follows:

$$\partial_\mu \tilde{F}^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu F_{\alpha\beta} = +\frac{1}{2}\epsilon^{\nu\mu\alpha\beta}\partial_\mu F_{\alpha\beta} \tag{E.367}$$

by substituting the definition of the duality transform and by interchanging $\mu \leftrightarrow \nu$ in the last step, which brings in a minus-sign because of the antisymmetry of ϵ. In fact, any cyclic permutation of the indices does not change anything, so that one can write

$$\ldots = \frac{1}{6}\left[\epsilon^{\nu\mu\alpha\beta} + \epsilon^{\nu\alpha\beta\mu} + \epsilon^{\nu\beta\mu\alpha}\right]\partial_\mu F_{\alpha\beta} = \frac{1}{6}\epsilon^{\nu\mu\alpha\beta}\underbrace{\left(\partial_\mu F_{\alpha\beta} + \partial_\beta F_{\mu\alpha} + \partial_\alpha F_{\beta\mu}\right)}_{=0} = 0 \tag{E.368}$$

making $\partial_\mu \tilde{F}^{\mu\nu} = 0$ equivalent with eqn. (E.365), after renaming the indices in the second and third term.

The Bianchi-identity is particularly interesting because it provides a propagation mechanism for electromagnetic waves: Acting on eqn. (E.365) with the derivative $\eta^{\mu\nu}\partial_\nu$ gives

$$\eta^{\mu\nu}\partial_\nu\left(\partial_\mu F_{\alpha\beta} + \partial_\beta F_{\mu\alpha} + \partial_\alpha F_{\beta\mu}\right) = \underbrace{\eta^{\mu\nu}\partial_\nu\partial_\mu}_{=\Box} F_{\alpha\beta} + \partial_\beta \underbrace{\eta^{\mu\nu}\partial_\nu F_{\mu\alpha}}_{=0} - \partial_\alpha \underbrace{\eta^{\mu\nu}\partial_\nu F_{\mu\beta}}_{=0} = 0, \tag{E.369}$$

and substituting the field equation for vacuum twice has us arrive at a wave equation for the fields,

$$\Box F_{\alpha\beta} = 0. \tag{E.370}$$

It can be solved with a wave ansatz $F_{\alpha\beta} \propto \exp(\pm i k_\mu x^\mu)$, leading to the null-condition

$$\eta^{\mu\nu}k_\mu k_\nu = 0 \quad \text{equivalent with} \quad \left(\frac{\omega}{c}\right)^2 - \gamma^{ij}k_i k_j = 0 \rightarrow \omega = \pm ck \tag{E.371}$$

such that group velocity $\mathrm{d}\omega/\mathrm{d}k$ and phase velocity ω/k are both c, and dispersion can not occur.

The wave equation for a non-vacuum situation looks a bit weird: Substituting the sources $\jmath^\alpha$ and $\jmath^\beta$ gives

$$\Box F_{\alpha\beta} = \frac{4\pi}{c}\left(\partial_\alpha \eta_{\beta\mu}\jmath^\mu - \partial_\beta \eta_{\alpha\mu}\jmath^\mu\right), \tag{E.372}$$

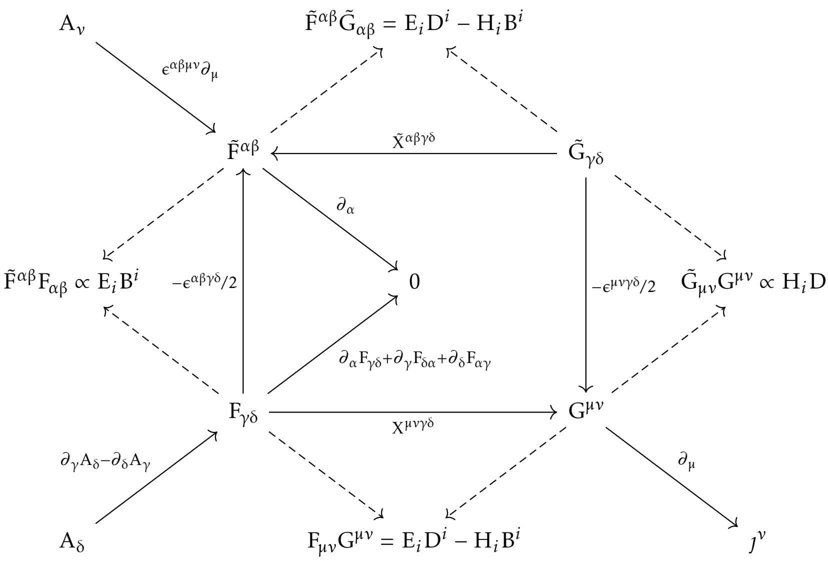
where it is interesting to see that the antisymmetry in the index pair (α, β) appears consistently in the sources on the right side. The same result could have been derived from the potentials, too, as $\Box A_\mu = 4\pi/c\ \eta_{\mu\nu}\jmath^\nu$ in e.g. Lorenz-gauge becomes

$$\Box F_{\alpha\beta} = \Box\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) = \partial_\alpha \Box A_\beta - \partial_\beta \Box A_\alpha = \frac{4\pi}{c}\left(\partial_\alpha \eta_{\beta\mu}\jmath^\mu - \partial_\beta \eta_{\alpha\mu}\jmath^\mu\right). \tag{E.373}$$

Actually, eqn. (E.373) is able to explain an interesting fact: Naively, one would think that it is not entirely clear how the six components of $G^{\mu\nu}$ are sourced by the four components of $\jmath^\mu$, and only going through the potential A_μ resolves the issue: There is, in particular in Lorenz-gauge (just for illustration, any gauging term $\partial_\mu\chi$ would drop from the expression), a one-to-one relation linking A_μ to $\jmath^\mu$ in $\Box A_\mu = 4\pi/c\,\eta_{\mu\nu}\jmath^\nu$, and the definition of $F_{\mu\nu}$ as $\partial_\mu A_\nu - \partial_\nu A_\mu$ then generates six mutually independent field components, to be related linearly to the six free components of $G^{\mu\nu}$ through the constitutive relation.

On the other hand, eqn. (E.373) may be interpreted in a way that it is not the current density $\jmath^\alpha$ that sources $F_{\alpha\beta}$, but rather its antisymmetric derivative $\partial_\alpha\eta_{\beta\mu}\jmath^\mu - \partial_\beta\eta_{\alpha\mu}\jmath^\mu$. Its six components determine each individually and independently the six components of $F_{\alpha\beta}$, even in a physical and gauge independent way.

A summary of the two field tensors and their duals, along with all four possible quadratic Lorentz-invariants (three of which are distinct, and reduce to two in vaccum) is given by this diagram:



(E.374)

E.5 *Covariant electrodynamics*

Summarising the results from the previous chapters shows that there is a clear conceptual picture defining Maxwell-electrodynamics:

	vector	*form*
⚠ *matter*	$G^{\mu\nu}$	$\tilde{G}_{\mu\nu}$
vacuum	$\tilde{F}^{\mu\nu}$	$F_{\mu\nu}$

- The 4-potential A_μ and the 4-current $\jmath^\mu$ are a Lorentz-linear form and a Lorentz-vector, respectively.
- The inhomogeneous Maxwell-equation take on the form $\partial_\mu G^{\mu\nu} = 4\pi/c\;\jmath^\nu$ and the homogeneous Maxwell-equations are written as $\partial_\mu\tilde{F}^{\mu\nu} = 0$, as there are no magnetic charges.
- Equivalent to the homogeneous Maxwell equation is the Bianchi-identity, $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$, which is automatically fulfilled if $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

- Charge is conserved and the inhomogeneous Maxwell-equation $\partial_\mu j^\mu = 0$ respects it through the antisymmetry of $G^{\mu\nu}$.

- Gauging with a gauge function χ implies the transformation $A_\mu \to A_\mu + \partial_\mu\chi$, leaving the Faraday tensor $F_{\mu\nu}$ invariant through its antisymmetry.

- Under the Lorenz-gauge condition $\eta^{\mu\nu}\partial_\mu A_\nu = 0$ one obtains a typical wave equation $\Box A_\mu = 4\pi/c\ \eta_{\mu\nu} j^\nu$ from the inhomogeneous Maxwell-equation, with Lorentz-invariant propagation speed c.

- The geometry is defined by the metric tensor $\eta_{\mu\nu}$ which is relevant for the vacuum fields in $F_{\mu\nu}$. The constitutive relation $X^{\alpha\beta\mu\nu}$ links $G^{\mu\nu}$ to $F_{\mu\nu}$ and falls back onto the metric in vacuum.

It is amazing to see how clearly gauge-transforms and Lorentz-transforms are incorporated into the formalism, and how the mathematical structure of the Maxwell-equations results from the antisymmetry of the field tensor, as well as its gauge-independence. It's worthwhile to contemplate, how the Lorenz-gauge condition $\eta^{\mu\nu}\partial_\mu A_\nu = 0$ is at the same time a Lorentz-invariant: As a Lorentz-scalar it has the same value, zero in this case, in all frames. The electromagnetic field, too, possesses Lorentz-invariants, which are necessarily quadratic or of higher order in the fields, as all contractions $F^\mu{}_\mu = \eta^{\mu\nu}F_{\mu\nu} = 0$, $\tilde{F}^\mu{}_\mu = \eta_{\mu\nu}\tilde{F}^{\mu\nu} = 0$, $\tilde{G}^\mu{}_\mu = \eta^{\mu\nu}\tilde{G}_{\mu\nu} = 0$ and lastly $G^\mu{}_\mu = \eta_{\mu\nu}G^{\mu\nu} = 0$ vanish because of the antisymmetry of $F_{\mu\nu}$, $G^{\mu\nu}$ and their respective duals.

Quadratic invariants are first of all

$$F_{\mu\nu}G^{\mu\nu} = \tilde{F}^{\mu\nu}\tilde{G}_{\mu\nu} = E_i D^i - H_i B^i, \tag{E.375}$$

which is a properly scalar quantity which is in addition parity-positive: The product of two parity-even magnetic fields is parity-even and the product of two parity-odd electric fields is likewise parity-even. Mixed contractions involving a single dual,

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = 4E_i B^i \quad \text{and} \quad \tilde{G}_{\mu\nu}G^{\mu\nu} = 4H_i D^i \tag{E.376}$$

are parity negative, as products of a parity-even magnetic field and a parity-odd electric field.

⚠ *The Maxwell-field has a single, scalar quadratic invariant, $F_{\mu\nu}G^{\mu\nu}$; there are two pseudoscalar quadratic invariants, $\tilde{F}^{\mu\nu}F_{\mu\nu}$ and $\tilde{G}_{\mu\nu}G^{\mu\nu}$, where the last two coincide in vacuum.*

In particular the first invariant does not reflect an energy density $T^{tt} \propto E_i D^i + H_i B^i$, which should depend on the choice of frame and can not be invariant. Its numerical value is actually zero for all vacuum solutions, as can be quickly verified by considering a plane wave: The electric and magnetic energy densities are equal at every point and instant, $E_i D^i = H_i B^i$, making sure that $F_{\mu\nu}G^{\mu\nu} = 0$. Furthermore, the electric and magnetic fields are orthogonal to each other, such that $E_i B^i = 0$ and $H_i D^i = 0$.

The invariant discussed above are contractions between the vectorial tensors $G^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ on one side and the linear forms $F_{\mu\nu}$ and $\tilde{G}_{\mu\nu}$ on the other. In a vacuum situation, all vectorial quantities are trivially related to their linear forms through the Minkowski-metric, so it is possible to construct 4 more invariants

$$\eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} = \eta^{\alpha\mu}\eta^{\beta\nu}\tilde{G}_{\alpha\beta}\tilde{G}_{\mu\nu} = \eta_{\alpha\mu}\eta_{\beta\nu}G^{\alpha\beta}G^{\mu\nu} = \eta_{\alpha\mu}\eta_{\beta\nu}\tilde{F}^{\alpha\beta}\tilde{F}^{\mu\nu} \propto \gamma^{ij}E_iE_j - \gamma_{ij}B^iB^j. \tag{E.377}$$

E.6 *Lagrange-density for the dynamics of fields*

To our knowledge, all fundamental physical theories can be derived from variational principles, and electrodynamics is no exception. At the basis of all variational principles is the notion that the action is invariant under a certain relativity principle, in our case Lorentz-relativity, which leads to a covariant equation of motion, where all quantities are consistently behaving under changes in the frame: This was already the case for Galilean dynamics, as a rotationally invariant Lagrange-function $\mathcal{L}(x^i, \dot{x}^i) = \gamma_{ij}\dot{x}^i\dot{x}^j/2 - \Phi(x^i)$ with the Euclidean, rotationally invariant scalar product $\gamma_{ij}\dot{x}^i\dot{x}^j$ gave rise to a equation of motion $\ddot{x}^i = -\gamma^{ij}\partial_j\Phi$ relating two vectors to each other. From this point of view one would hope to arrive at a Lorentz-covariant equation of motion from a Lorentz-invariant Lagrange function. As the Euler-Lagrange-equation usually reduces the powers by one in the derivative process, one would like to begin with quadratic Lorentz-invariants in order to arrive at a linear field equation which respects the superposition principle. Then, if the Lagrange-function does not depend explicitly on the coordinates x^μ, i.e. if x^μ is a cyclic variable, one has reasons to expect that the theory is conserving energy and momentum. And lastly, charge conservation should result from gauge-invariance as the symmetry principle.

⚠ *invariance/covariance principle: covariant field equations from invariant Lagrange functions*

⚠ *Quadratic Lagrange functions lead to linear field equations: superposition principle*

⚠ *coordinates as cyclic variables imply energy-momentum conservation*

⚠ *gauge symmetry is related to charge conservation*

E.6.1 Scalar field on a Euclidean background

Let's illustrate how variational principles work with a simpler example than the full Maxwell-theory. Electrostatics is fully characterised by a potential Φ which is linked to the source ρ by means of the Poisson-equation $\Delta\Phi = -4\pi\rho$, in other words: We're looking for a variational principle for a scalar field φ on a Euclidean background, that is coupled to a source and does not have any dynamics on its own. Writing the action S as an integral over a Lagrange-density $\mathcal{L}$ would give

$$S = \int_V d^3x\, \mathcal{L}(\varphi, \partial_i\varphi) \tag{E.378}$$

and Hamilton's principle $\delta S = 0$ then suggests the variation

$$\delta S = \delta\int_V d^3x\,\mathcal{L} = \int_V d^3x\left(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\delta\partial_i\varphi\right) \tag{E.379}$$

Interchanging the partial derivative and the variation, $\delta\partial_i\varphi = \partial_i\delta\varphi$, allows an integration by parts. One can isolate the Euler-Lagrange-equation for a scalar field φ

$$\delta S = \int_V d^3x\left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\right)\delta\varphi = 0 \quad\rightarrow\quad \partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} \tag{E.380}$$

because the variation $\delta\varphi$ is zero by construction on the boundary ∂V,

$$\int_V dV\,\partial_i\left(\frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\delta\varphi\right) = \int_{\partial V} dS_i\left(\frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\delta\varphi\right) = 0 \quad\text{as}\quad \frac{\partial\mathcal{L}}{\partial\partial_i\varphi}\delta\varphi\Big|_{\partial V} = 0. \tag{E.381}$$

The Poisson-equation as a second order partial differential equation should result from an action that contains squares of first derivatives of the potential φ, for instance

$$\mathcal{L}(\varphi, \partial_i\varphi) = \frac{\gamma^{ab}}{2}\partial_a\varphi\partial_b\varphi - 4\pi\rho\varphi. \tag{E.382}$$

Concerning the invariance-covariance principle, we note that the first term is as a scalar product, invariant under rotations. Substitution into the Euler-Lagrange equation gives

$$\frac{\partial\mathcal{L}}{\partial\varphi} = -4\pi\rho \tag{E.383}$$

as well as (please always rename the indices when you're doing this)

$$\frac{\partial\mathcal{L}}{\partial\partial_i\varphi} = \frac{\gamma^{ab}}{2}\left(\frac{\partial\partial_a\varphi}{\partial\partial_i\varphi}\partial_b\varphi + \partial_a\varphi\frac{\partial\partial_b\varphi}{\partial\partial_i\varphi}\right) =$$

$$\frac{\gamma^{ab}}{2}\left(\delta^i_a\partial_b\varphi + \partial_a\varphi\delta^i_b\right) = \frac{1}{2}\left(\gamma^{ib}\partial_b\varphi + \gamma^{ai}\partial_a\varphi\right) = \gamma^{ib}\partial_b\varphi \tag{E.384}$$

such that one arrives precisely at the Poisson-equation

$$\partial_i\frac{\partial\mathcal{L}}{\partial\partial_i\varphi} = \partial_i\gamma^{ib}\partial_b\varphi = \Delta\varphi = \frac{\partial\mathcal{L}}{\partial\varphi} = -4\pi\rho. \tag{E.385}$$

where the Laplace-operator Δ is scalar and does not change under rotations.

E.6.2 Scalar field on a Lorentz background

Repeating the entire derivation for a relativistic field theory with the Lagrange density

$$\mathcal{L}(\varphi, \partial_\mu\varphi) = \frac{\eta^{\mu\nu}}{2}\partial_\mu\varphi\partial_\nu\varphi + 4\pi\rho\varphi \tag{E.386}$$

leads with the Euler-Lagrange equation

$$\partial_\alpha\frac{\partial\mathcal{L}}{\partial\partial_\alpha\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} \tag{E.387}$$

for varying the action

$$S = \int_V d^4x\; \mathcal{L}(\varphi, \partial_\mu\varphi) \tag{E.388}$$

that results as an integral over the spacetime volume $d^4x = c\,dt\,d^3x$. Carrying out the variation $\delta S = 0$ implies the wave equation

$$\Box\varphi = 4\pi\rho \quad \text{with} \quad \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu. \tag{E.389}$$

as a generalisation to the Poisson-equation.

E.6.3 Maxwell field on a Lorentz background

The Maxwell-equations expressed in terms of the potential A_μ are likewise second order differential equations, where the action should contain squares of first derivatives of the potential. The new aspect now is that the potential has (4) internal degrees of freedom and is not scalar as in the previous two examples. The squares of the first derviatives of A_μ should be Lorentz-invariants, and we will only utilise the parity-positive one for the time being.

Driven by analogy, one would write for a vacuum situation

⚠ please keep in mind that the Lagrange-density is invariant under affine transforms, $\mathcal{L} \rightarrow \alpha\mathcal{L}+\beta$, therefore only the ratio of prefactors matters.

$$S = \int_V d^4x\, \mathcal{L}(A_\mu, \partial_\mu A_\nu) = \int_V d^4x \quad \underbrace{\eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}F_{\mu\nu}}_{\text{square of first derivatives}} \quad + \quad \underbrace{\frac{16\pi}{c}A_\mu\jmath^\mu}_{\text{coupling to the source}} \tag{E.390}$$

Please keep in mind that it is only through broken duality and the non-existence of magnetic charges that the potentials A_μ exist such which ultimately enables a Lagrangian description as in eqn. (E.390). A suitable Euler-Lagrange equation would result from variation δS of the action S with respect to δA, which becomes

$$\delta S = \delta\int_V d^4x\,\mathcal{L} = \int_V d^4x\left(\frac{\partial\mathcal{L}}{\partial A_\gamma}\delta A_\gamma + \frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}\delta\partial_\gamma A_\delta\right) = \int_V d^4x\left(\frac{\partial\mathcal{L}}{\partial A_\delta} - \partial_\gamma\frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}\right)\delta A_\delta = 0 \tag{E.391}$$

where as always we wrote $\delta\partial_\gamma A_\delta = \partial_\gamma\delta A_\delta$ for the integration by parts, finally allowing the extraction of the Euler-Lagrange equation by means of Hamilton's principle $\delta S = 0$:

$$\partial_\gamma\frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta} = \frac{\partial\mathcal{L}}{\partial A_\delta}, \tag{E.392}$$

again keeping the variation δA_δ fixed on the boundary,

$$\int_V dV\,\partial_\gamma\left(\frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}\delta A_\delta\right) = \int_{\partial V} dS_\gamma\left(\frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}\delta A_\delta\right) = 0 \quad \text{as} \quad \frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}\delta A_\delta\Big|_{\partial V} = 0. \tag{E.393}$$

Substitution of the Lagrange-density $\mathcal{L}$ is rather straightforward for the ∂A_δ-derivative,

$$\frac{\partial\mathcal{L}}{\partial A_\delta} = \frac{16\pi}{c}\frac{\partial A_\mu}{\partial A_\delta}\jmath^\mu = \frac{16\pi}{c}\delta^\delta_\mu\jmath^\mu = \frac{16\pi}{c}\jmath^\delta \tag{E.394}$$

but involves handling many indices for the derivatives with respect to $\partial_\gamma A_\delta$.

Instead, one can rewrite the derivative as

$$\frac{\partial}{\partial\partial_\gamma A_\delta} = \frac{\partial F_{\sigma\tau}}{\partial\partial_\gamma A_\delta}\frac{\partial}{\partial F_{\sigma\tau}} = \frac{\partial(\partial_\sigma A_\tau - \partial_\tau A_\sigma)}{\partial\partial_\gamma A_\delta}\frac{\partial}{\partial F_{\sigma\tau}} = \left(\frac{\partial\partial_\sigma A_\tau}{\partial\partial_\gamma A_\delta} - \frac{\partial\partial_\tau A_\sigma}{\partial\partial_\gamma A_\delta}\right)\frac{\partial}{\partial F_{\sigma\tau}} =$$
$$\left(\delta_\sigma^\gamma\delta_\tau^\delta - \delta_\tau^\gamma\delta_\sigma^\delta\right)\frac{\partial}{\partial F_{\sigma\tau}} = \frac{\partial}{\partial F_{\gamma\delta}} - \frac{\partial}{\partial F_{\delta\gamma}} = 2\frac{\partial}{\partial F_{\gamma\delta}}. \quad \text{(E.395)}$$

In both cases, the elementary derivatives give either 0 or 1 according to

$$\frac{\partial\partial_\mu A_\nu}{\partial\partial_\gamma A_\delta} = \delta_\mu^\gamma\delta_\nu^\delta \quad \text{as well as} \quad \frac{\partial A_\mu}{\partial A_\gamma} = \delta_\mu^\gamma, \quad \text{(E.396)}$$

because the field components and their derivatives into the different coordinate directions are all independent. The derivatives $\partial F_{\alpha\beta}/\partial F_{\mu\nu}$ of the field tensor with respect to itself are slightly more involved, because of the antisymmetry of both $F_{\alpha\beta}$ and $F_{\mu\nu}$. The necessary (anti-)symmetrisation reads

$$\frac{\partial F_{\alpha\beta}}{\partial F_{\mu\nu}} = \frac{1}{4}\left(\delta_\alpha^\mu\delta_\beta^\nu - \delta_\alpha^\nu\delta_\beta^\mu - \delta_\beta^\mu\delta_\alpha^\nu + \delta_\beta^\nu\delta_\alpha^\mu\right) = \frac{1}{2}\left(\delta_\alpha^\mu\delta_\beta^\nu - \delta_\alpha^\nu\delta_\beta^\mu\right) \quad \text{(E.397)}$$

with a simplification due to the pairwise identity of terms.

Then, application of the differentiations to the kinetic term required by the Euler-Lagrange equation yields:

$$\frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta} = 2\frac{\partial}{\partial F_{\gamma\delta}}\eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} = 2\eta^{\alpha\mu}\eta^{\beta\nu}\left(\frac{\partial F_{\alpha\beta}}{\partial F_{\gamma\delta}}F_{\mu\nu} + F_{\alpha\beta}\frac{\partial F_{\mu\nu}}{\partial F_{\gamma\delta}}\right) =$$
$$\eta^{\alpha\mu}\eta^{\beta\nu}\left(\left(\delta_\alpha^\gamma\delta_\beta^\delta - \delta_\alpha^\delta\delta_\beta^\gamma\right)F_{\mu\nu} + F_{\alpha\beta}\left(\delta_\mu^\gamma\delta_\nu^\delta - \delta_\mu^\delta\delta_\nu^\gamma\right)\right) = 4\eta^{\gamma\mu}\eta^{\delta\nu}F_{\mu\nu}. \quad \text{(E.398)}$$

Collection of all results suggests as the field equation the relation

$$\partial_\gamma\frac{\partial\mathcal{L}}{\partial_\gamma A_\delta} = 4\partial_\gamma\eta^{\gamma\mu}\eta^{\delta\nu}F_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial A_\delta} = \frac{16\pi}{c}\jmath^\delta \quad \rightarrow \quad \eta^{\gamma\mu}\partial_\gamma F_{\mu\nu} = \frac{4\pi}{c}\eta_{\delta\nu}\jmath^\delta \quad \text{(E.399)}$$

which one immediately recognises as the inhomogeneous Maxwell-equation in vacuum, with the divergence of the field tensor being equated to the source. The invariance of the Lagrangian description and the covariance of the field equation is summarised by this diagram,

$$\begin{array}{cccc} S = \int_V d^4x \quad \eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} & + & \frac{16\pi}{c}A_\mu\jmath^\mu & \\ \downarrow \delta S=0 & & \downarrow \delta S=0 & \\ \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\mu F_{\alpha\beta} & - & \frac{4\pi}{c}\jmath^\nu & = 0, \end{array} \quad \text{(E.400)}$$

and substitution of the expression for $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ finally leads to

$$\eta^{\gamma\mu}\partial_\gamma F_{\mu\nu} = \eta^{\gamma\mu}\partial_\gamma\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) = \underbrace{\eta^{\gamma\mu}\partial_\gamma\partial_\mu A_\nu}_{=\Box A_\nu} - \partial_\nu \underbrace{\eta^{\gamma\mu}\partial_\gamma A_\mu}_{=0} = \frac{4\pi}{c}\eta_{\delta\nu}\jmath^\delta, \tag{E.401}$$

which clearly demonstrates a covariant wave equation with the potential A_ν as a linear form related to the source $\eta_{\delta\nu}\jmath^\delta$, a vector converted into a linear form, with the assumption of Lorenz-gauge $\eta^{\gamma\mu}\partial_\gamma A_\mu = 0$ for making the second term disappear.

Formal application of the variation to the action integral would be an expression

$$\delta S = \delta \int_V d^4x\, \eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} = 2\int_V d^4x\, \eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}\,\delta F_{\mu\nu} = 0 \tag{E.402}$$

where one can interpret the requirement of Hamilton's principle, namely $\delta S = 0$, as an orthogonality condition between $F_{\alpha\beta}$ and its variation $\delta F_{\alpha\beta}$, as a modern embodiment of the principle of virtual work.

It might be an interesting endeavour to understand how exactly the structure $\eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}F_{\mu\nu}$ in the kinetic term of the Lagrange density is to be interpreted, beyond the fact that it is a quadratic Lorentz-invariant. With the antisymmetry of $F_{\mu\nu} = -F_{\nu\mu}$ one can write

$$S = \int_V d^4x\, \eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} = \int_V d^4x\, \eta^{\alpha\mu}\eta^{\beta\nu}\frac{1}{2}\left(F_{\alpha\beta}F_{\mu\nu} - F_{\alpha\beta}F_{\nu\mu}\right) \tag{E.403}$$

which becomes, after renaming the indices $\mu \leftrightarrow \nu$ in the second term,

$$S = \frac{1}{2}\int_V d^4x\, \eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}F_{\mu\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}F_{\alpha\beta}F_{\mu\nu} = \int_V d^4x\, \frac{\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}}{2}F_{\alpha\beta}F_{\mu\nu} \tag{E.404}$$

which can be written as

$$S = \int_V d^4x\, X^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \quad \text{with a measure} \quad X^{\alpha\beta\mu\nu} = \frac{\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}}{2} \tag{E.405}$$

as tensor with two antisymmetric index pairs (α,μ) and (β,ν). Perhaps the index structure reminds you of the Grassmann-relation $\gamma_{il}\epsilon^{ijk}\epsilon^{lmn} = \gamma^{jm}\gamma^{kn} - \gamma^{jn}\gamma^{km}$ of a square of a vector product, which quantifies the area spanned by two vectors: In some sense, the same happens in the Lagrange density, which is an abstract measure of the area between ∂_μ and A_ν, induced by the metric $\eta^{\mu\nu}$.

E.6.4 Maxwell field in matter

For the behaviour of the Maxwell field in matter a suitable starting point could be the action

$$S = \int_V d^4x\, F_{\mu\nu}G^{\mu\nu} + \frac{16\pi}{c}A_\mu \jmath^\mu \tag{E.406}$$

where the Lorentz invariant in matter constitutes the kinetic term. Expressed in terms of the fields it reads $F_{\mu\nu}G^{\mu\nu} = E_i D^i - H_i B^i$. On possible pathway to carry out the variation and to perform the derivatives with respect to A_γ and $\partial_\gamma A_\delta$ is provided by the constitutive relation,

$$G^{\alpha\beta} = X^{\alpha\beta\mu\nu} F_{\mu\nu}, \tag{E.407}$$

that relates the fields D^i and H_i contained in $G^{\mu\nu}$ to the vacuum fields E_i and B^i in $F_{\mu\nu}$. After all, only $F_{\mu\nu}$ follows from the derivation of the potential A_μ and is accessible to variation. As both tensors are antisymmetric, $X^{\alpha\beta\mu\nu}$ has to be antisymmetric in each index pair, $X^{\alpha\beta\mu\nu} = -X^{\alpha\beta\nu\mu} = -X^{\beta\alpha\mu\nu} = X^{\beta\alpha\nu\mu}$. Then, the action integral reads

$$S = \int_V d^4x \; X^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} + \frac{16\pi}{c} A_\mu \jmath^\mu \tag{E.408}$$

Variation proceeds as in the previous case, as

$$\frac{\partial \mathcal{L}}{\partial \partial_\gamma A_\delta} = 2 \frac{\partial}{\partial F_{\gamma\delta}} X^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} = 2 X^{\alpha\beta\mu\nu} \left(\frac{\partial F_{\alpha\beta}}{\partial F_{\gamma\delta}} F_{\mu\nu} + F_{\alpha\beta} \frac{\partial F_{\mu\nu}}{\partial F_{\gamma\delta}} \right) =$$
$$X^{\alpha\beta\mu\nu} \left(\left(\delta^\gamma_\alpha \delta^\delta_\beta - \delta^\delta_\alpha \delta^\gamma_\beta \right) F_{\mu\nu} + F_{\alpha\beta} \left(\delta^\gamma_\mu \delta^\delta_\nu - \delta^\delta_\mu \delta^\gamma_\nu \right) \right) = 4 X^{\gamma\delta\mu\nu} F_{\mu\nu} = 4 G^{\gamma\delta}. \tag{E.409}$$

Combined with the previous result on the derivative with respect to A_δ, the Euler-Lagrange equation yields:

$$\partial_\gamma \frac{\partial \mathcal{L}}{\partial_\gamma A_\delta} = 4 \partial_\gamma X^{\gamma\delta\mu\nu} F_{\mu\nu} = 4 \partial_\gamma G^{\gamma\delta} = \frac{\partial \mathcal{L}}{\partial A_\delta} = \frac{16\pi}{c} \jmath^\delta \quad \rightarrow \quad \partial_\gamma G^{\gamma\delta} = \frac{4\pi}{c} \jmath^\delta, \tag{E.410}$$

which is in fact the Maxwell field equation in matter. While the Lagrange density eqn. (E.406) is the source of the field equation and links ultimately of the fields D^i and H_i to the sources, the dynamics of the dual field tensor $\tilde{F}^{\mu\nu}$ with E_i and B^i is already fixed by the Bianchi-identity.

E.7 *Optics*

It is fair to say that the covariant constitutive relation falls back in isotropic media on the antisymmetrised metric,

$$X^{\alpha\beta\mu\nu} = \frac{\eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}}{2} \tag{E.411}$$

possibly with $(\epsilon\mu)$ as a prefactor in isotropic media in the spatial part of the metric. In fact, in isotropic media one gets for the effective metric

⚠ *Please keep in mind that $D^i = \epsilon^{ij} E_j \propto \epsilon\gamma^{ij} E_j$ and $H_i = \mu_{ij} B^j \propto \mu\gamma_{ij} B^j$, so the mapping from the vacuum fields E_i, B^i in $F_{\mu\nu}$ to D^i, H_i in $G^{\mu\nu}$ picks up a factor of $\epsilon\mu = n^2$ in the spatial components.*

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -n^2 & & \\ & & -n^2 & \\ & & & -n^2 \end{pmatrix} \quad \leftrightarrow \quad \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -n^{-2} & & \\ & & -n^{-2} & \\ & & & -n^{-2} \end{pmatrix} \tag{E.412}$$

with the refractive index $n = \sqrt{\epsilon\mu}$.

In this particular case, a plane-wave ansatz $\exp(\pm\mathrm{i}k_\alpha x^\alpha)$ yields a modified null-condition

$$\eta^{\mu\nu}k_\mu k_\nu = 0 = \left(\frac{\omega}{c}\right)^2 - \frac{k^2}{n^2} \quad \rightarrow \quad \omega = \pm\frac{ck}{n} \tag{E.413}$$

Consequently, the velocities are diminished by the refractive index n,

$$\upsilon_{\mathrm{gr}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{c}{n} = \frac{\omega}{k} = \upsilon_{\mathrm{ph}} \tag{E.414}$$

and the light cone becomes narrower by the factor n. As constitutive tensor $\mathrm{X}^{\alpha\beta\mu\nu}$ is composed of the two contributions, namely the permissivity tensor ϵ^{ij} and the permeability tensor μ^{ij}, on the spatial components are affected: This effectively means that in a medium, the wave length $\lambda = 2\pi/k$ is affected by the refractive index and not the angular frequency ω.

The notion, that wave length changes in a medium according to $\lambda \rightarrow n\lambda$ with the refractive index n, paving the way for Fermat's principle for refraction: The optical path length is effectively increased by the same factor of n. The *spatial* distance between two point A and B is given by

$$s = \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}s \quad \rightarrow \quad \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}s\, n = \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{d}\lambda \sqrt{\gamma_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\, n(x^i) \tag{E.415}$$

and is extremised according to $\delta s = 0$ to yield the actual light path, technically through application of the Euler-Lagrange equation albeit for a rather unusual form of the Lagrange-function

$$\mathcal{L} = \sqrt{\gamma_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}}\, n(x^i) \tag{E.416}$$

with no additive separation in a kinetic and potential part. Instead, in applying the Euler-Lagrange equation (abbreviating $\dot{x}^i = \mathrm{d}x^i/\mathrm{d}\lambda$)

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial\mathcal{L}}{\partial\dot{x}^a} = \frac{\partial\mathcal{L}}{\partial x^a} \tag{E.417}$$

one needs to be careful because after the $\partial\dot{x}^i$-differentiation, $\mathcal{L}$ still depends on x^i, which yields additional terms involving $\dot{x}^i$ in the $\mathrm{d}\lambda$-differentiation, in particular the gradient of the refractive index $\mathrm{d}n/\mathrm{d}\lambda = \partial_a n\,\dot{x}^a$. The first two derivatives are

$$\frac{\partial\mathcal{L}}{\partial x^a} = \sqrt{\gamma_{ij}\dot{x}^i\dot{x}^j}\,\partial_a n, \quad \text{followed by} \quad \frac{\partial\mathcal{L}}{\partial\dot{x}^a} = \frac{n\gamma_{ai}\dot{x}^i}{\sqrt{\gamma_{mn}\dot{x}^m\dot{x}^n}}, \tag{E.418}$$

but increase dramatically in their complexity in the $\mathrm{d}\lambda$-differentiaton. Ultimately, these equations lead to the concept of Lagrangian optics and can only be solved sensibly either through numerical methods or in approximations. While we commonly assumed homogeneous media, the formalism is still applicable in the limit of geometric optics where the scale on which n changes is large compared to the scale on which the fields vary, i.e. the wave length λ.

While it is clear that the metric in an anisotropic medium can show different light propagation speeds along the three coordinate directions, the constitutive tensor $X^{\alpha\beta\mu\nu}$: The wave equation in the most general case reads

$$\partial_\alpha G^{\alpha\beta} = X^{\alpha\beta\mu\nu}\partial_\alpha F_{\mu\nu} = 2X^{\alpha\beta\mu\nu}\partial_\alpha\partial_\mu A_\nu = 0, \tag{E.419}$$

which suggest for an ansatz $A_\mu \propto A_\mu^{(0)}\exp(\pm ik_\gamma x^\gamma)$, with an amplitude $A_\mu^{(0)}$ that contains information about polarisation. Then, the null-condition reads

$$X^{\alpha\beta\mu\nu}A_\nu^{(0)}\, k_\alpha k_\mu = 0 \tag{E.420}$$

and is effectively a polarisation-dependent dispersion relation, with differences in propagation speeds even into the same direction for different polarisations: This phenomenon is known as birefringence, and can be observed in e.g. calcite crystals.

⚠ *Please note how the null-condition requires a summation over the pair (α, μ) and* not (μ, ν) *which would be trivially zero.*

E.8 *Gauge-invariance and charge conservation*

Gauge-invariance of the term $\eta^{\alpha\mu}\eta^{\beta\nu}F_{\alpha\beta}F_{\mu\nu}$ is clearly given, as $F_{\mu\nu}$ does not change under gauge-transformation anyways. But it is interesting to see how gauge-invariance is recovered in the entire Lagrange-formalism. In fact, with $A_\mu \to A_\mu + \partial_\mu\chi$ one obtains

$$S = \int_V d^4x\, \mathcal{L} \to S + \frac{16\pi}{c}\int_V d^4x\, \partial_\mu\chi\, \jmath^\mu = S + \frac{16\pi}{c}\int_V d^4x\, \left[\partial_\mu(\chi\jmath^\mu) - \chi\partial_\mu\jmath^\mu\right] \tag{E.421}$$

where $\partial_\mu\jmath^\mu = 0$ due to continuity of the charge density. The first term can be converted into a surface integral with the Gauß-theorem,

$$S \to S + \frac{16\pi}{c}\int_{\partial V} dS_\mu\, (\chi\jmath^\mu) = S \tag{E.422}$$

i.e. one recovers gauge invariance when assuming a localised charge distribution: moving the integration surface ∂V out leads to $\chi\jmath^\mu$ vanishing faster than ∂V increases, and consequently, the integral approaches zero. Hence, the action is gauge invariant if charge is conserved. To show the opposite is impossible for our current understanding of charge as a source of the electromagnetic field and requires a more detailed model for the charge-carrying matter in the form of a quantum theory.

⚠ *Please note that there are different concepts at play to have terms vanish in S (locality of the charge distribution) and in δS (fixed variation on boundary).*

E.9 *Conservation of energy and momentum*

E.9.1 Scalar field on a Lorentz background

The Lagrange-density of the electromagnetic field does not depend explicitly on the coordinates x^μ, meaning that it is truly universal: The way in which the field is coupled to its charges and the internal dynamics is the same everywhere and at every time. As a consequence of the translation invariance along the ct- and x^i-coordinates, energy and momentum are conserved, which we should derive first for a scalar field φ. There, the Lagrange-density is given by $\mathcal{L}(\varphi, \partial_\mu\varphi)$ but *not* by $\mathcal{L}(\phi, \partial_\mu\varphi, x^\mu)$. The Euler-Lagrange equations follow from the variation of the action S

$$S = \int_V d^4x\, \mathcal{L}(\varphi, \partial_\mu\varphi) \quad \rightarrow \quad \delta S = \int_V d^4x \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right)\delta\varphi = 0 \tag{E.423}$$

such that Hamilton's principle $\delta S = 0$ implies

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} = \frac{\partial\mathcal{L}}{\partial\varphi} \tag{E.424}$$

If the Lagrange density $\mathcal{L}$ depends only on the fields themselves and not on the position, meaning the functional principle of the field theory as defined by $\mathcal{L}$ is the same everywhere and at very time, there is only one way in which the Lagrange density can change is moving through spacetime to a new point where the fields and their derivatives are different: The fields themselves need to change. This implies that under an infinitesimal shift in the coordinates into the direction ϵ^μ,

$$x^\mu \rightarrow x^\mu + \epsilon^\mu, \tag{E.425}$$

one expects a variation of the field $\delta\varphi$ to be

$$\delta\varphi = \varphi(x^\mu + \epsilon^\mu) - \varphi(x^\mu) = \epsilon^\alpha \partial_\alpha \varphi \tag{E.426}$$

and the corresponding variation of the Lagrange density would become

$$\delta\mathcal{L} = \epsilon^\alpha \partial_\alpha \mathcal{L} \tag{E.427}$$

On the other hand, the variation of the Lagrange density is given by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\partial_\mu\varphi = \left(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\right)\delta\varphi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi\right) \tag{E.428}$$

using the Leibnitz-rule. As the physical fields fulfil the Euler-Lagrange equation in the first term, only the second term remains, implying

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi\right) \tag{E.429}$$

Assembling the final expression from the variation $\delta\mathcal{L}$ in eqn. (E.429) with the expression eqn. (E.427) and the variation $\delta\varphi$ in eqn. (E.426) leads to

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\delta\varphi\right) - \epsilon^\alpha\partial_\alpha\mathcal{L} = 0 \tag{E.430}$$

such that, using $\partial_\alpha = \delta^\mu_\alpha \partial_\mu$,

$$\epsilon^\alpha\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi}\partial_\alpha\varphi - \delta^\mu_\alpha\mathcal{L}\right) = 0 \tag{E.431}$$

implying that there is a covariant divergence which vanishes,

$$\partial_\mu T_\alpha{}^\mu = 0 \tag{E.432}$$

with the energy-momentum tensor $T_\alpha{}^\mu$

$$T_\alpha{}^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \partial_\alpha \varphi - \delta^\mu_\alpha \mathcal{L}. \tag{E.433}$$

Effectively, this suggests a multidimensional Legendre-transform with the canonical field momentum π^μ

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \quad \text{such that} \quad T_\alpha{}^\mu = \pi^\mu\, \partial_\alpha \varphi - \delta^\mu_\alpha \mathcal{L}(\varphi, \pi^\mu) \tag{E.434}$$

where the structural similarity to the relation $\mathcal{H} = p_i \dot{x}^i - \mathcal{L}$ from classical mechanics is quite apparent.

If the Lagrange density had an additional dependence on the coordinates x^μ, it's variation (E.428) when transitioning form x^μ to $x^\mu + \epsilon^\mu$ would not only be caused by the different field amplitudes and their derivatives, but there would be a new term Q_α,

$$\delta\mathcal{L} = \epsilon^\alpha \partial_\alpha \mathcal{L}(\text{field variation}) + \epsilon^\alpha Q_\alpha(\text{explicit coordinate dependence}) \tag{E.435}$$

where this new term is effectively a source term to the otherwise vanishing continuity equation,

$$\partial_\mu T_\alpha{}^\mu = Q_\alpha. \tag{E.436}$$

The identification of $T_\alpha{}^\mu$ with the energy-momentum tensor becomes sensible for the case of a standard Lagrange-density for a scalar field φ,

$$\mathcal{L}(\varphi, \partial_\mu \varphi) = \frac{\eta^{\mu\nu}}{2} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \tag{E.437}$$

with a self-interaction potential $V(\varphi)$ that would contain e.g. a coupling to sources. Variation by substitution into the Euler-Lagrange equation yields directly the Klein-Gordon equation

$$\Box\varphi = -\frac{\partial V}{\partial \varphi} \quad \text{because} \quad \pi^\mu = \frac{\partial \mathcal{L}}{\partial_\mu \varphi} = \eta^{\mu\nu} \partial_\nu \varphi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \varphi} = -\frac{\partial V}{\partial \varphi} \tag{E.438}$$

with the next differentiation generating $\Box\varphi = \partial_\mu \pi^\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu \varphi$. Then, the tensor $T^\mu{}_\alpha$ becomes

$$T_\alpha{}^\mu = \pi^\mu \partial_\alpha \varphi - \delta^\mu_\alpha \mathcal{L}(\varphi, \pi^\mu) = \eta^{\mu\nu} \partial_\nu \varphi \partial_\alpha \varphi - \delta^\mu_\alpha \frac{\eta^{\gamma\delta}}{2} \partial_\gamma \varphi \partial_\delta \varphi + \delta^\mu_\alpha V(\varphi) \tag{E.439}$$

with the sign change in front of $V(\varphi)$ which is typical for the Legendre transform.

E.9.2 Maxwell field on a Lorentz background

There is a very important detail in the derivation of the energy-momentum tensor of the electromagnetic field, which otherwise proceeds exactly as in the case of the scalar field φ: When shifting the potential to compute δA_δ one should not use the derivative $\partial_\alpha A_\delta$ for forming $\delta A_\delta = \epsilon^\alpha \partial_\alpha A_\delta$ because it is not gauge-invariant. Rather, the variation should be given by the antisymmetrised form,

$$\delta A_\delta = \epsilon^\alpha \partial_\alpha A_\delta \rightarrow \epsilon^\alpha \left(\partial_\alpha A_\delta - \partial_\delta A_\alpha\right) = \epsilon^\alpha F_{\alpha\delta} \tag{E.440}$$

as the Faraday tensor $F_{\alpha\delta}$ is the gauge-invariant derivative of A_δ . The variation in the Lagrange-density becomes formally

watch out for gauge-independence in the derivative

$$\delta\mathcal{L} = \epsilon^\alpha \partial_\alpha \mathcal{L} \tag{E.441}$$

but expressed in terms of the fields, by virtue of the Leibnitz-rule,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial A_\delta}\delta A_\delta + \frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}\delta\partial_\gamma A_\delta = \left(\frac{\partial\mathcal{L}}{\partial A_\delta} - \frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}\right)\delta A_\delta + \partial_\gamma\left(\frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}\delta A_\delta\right), \tag{E.442}$$

where the first bracket disappears as it fulfils the Euler-Lagrange equation, that appears after the usual replacement $\delta\partial_\gamma A_\delta = \partial_\gamma \delta A_\delta$. The divergence in the second term can be reformulated as

$$\partial_\gamma\left(\frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}\delta A_\delta\right) = \epsilon^\alpha\partial_\gamma\left(\frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}F_{\alpha\delta}\right) = \delta\mathcal{L} = \epsilon^\alpha\partial_\alpha\mathcal{L} = \epsilon^\alpha\delta^\gamma_\alpha\partial_\gamma\mathcal{L} = \epsilon^\alpha\partial_\gamma\delta^\gamma_\alpha\mathcal{L} \tag{E.443}$$

so that the combination of the second and the sixth term suggest, as the shift ϵ^α was arbitrary:

$$\partial_\gamma\left(\frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}F_{\alpha\delta} - \delta^\gamma_\alpha\mathcal{L}\right) = 0, \tag{E.444}$$

i.e. a conservation law for the energy momentum tensor,

$$\partial_\gamma T_\alpha{}^\gamma = 0, \quad \text{with} \quad T_\alpha{}^\gamma = \frac{\partial\mathcal{L}}{\partial\partial_\gamma A_\delta}F_{\alpha\delta} - \delta^\gamma_\alpha\mathcal{L}. \tag{E.445}$$

The energy-momentum tensor $T_\mu{}^\nu$ is the relativistic generalisation of the Maxwell-tensor $T_i{}^j$, which makes up the spatial part of it. In vacuum, it is symmetric, $T_\mu{}^\nu = T_\nu{}^\mu$ and traceless, $T_\mu{}^\mu = \eta^{\mu\nu}T_{\mu\nu} = 0$: The physical meaning of this is not straightforward to understand, but essentially corresponds to the fact that there is no mass associated with the photons, i.e with excitations of the electromagnetic field. The components of $T_\mu{}^\nu$ contain the energy density, $T_t{}^t = E_i D^i - H_i B^i = w_\mathrm{el} + w_\mathrm{mag}$ and the Poynting-vector, $4\pi/c\, P^i = T_t{}^i$. In particular, the formulation of the Poynting-law would become $\partial_\mu T_t{}^\mu = \partial_t(w_\mathrm{el} + w_\mathrm{mag}) + \partial_i P^i = 0$.

Perhaps it's a weird and funny thought that Kirchhoff's mesh and knot rules for electric circuits are essentially reflections of the coordinate-independence of the Lagrange-function $\mathcal{L}$ giving rise to energy conservation, and of the gauge invariance of $\mathcal{L}$ compatible with charge conservation. And as a last remark in this context I would

		$\mathcal{C}$	$\mathcal{PT}$	$\mathcal{CPT}$
derivative	∂_μ	$+$	$-$	$-$
electric 4-current	$\jmath^\mu$	$-$	$+$	$-$
magnetic 4-current	$\imath^\mu$	$-$	$-$	$+$
Faraday tensor	$F_{\mu\nu}$	$-$	$+$	$-$
field tensor	$\tilde{F}^{\mu\nu}$	$-$	$-$	$+$

Table 2: Summary of the behaviour of all fields and sources in extended electrodynamics with electric and magnetic sources.

like to add that the construction with the infinitesimal shift of the Lagrange-density is in some sense a trick: Actually, one would like to construct a gradient $\partial\mathcal{L}$ of $\mathcal{L}$ which is caused by the fact that the fields and their derivatives have gradients. But one usually works with the convention that partial derivatives of functionals only apply to their explicit dependence on the coordinates, not their "indirect" position-dependence through the fields (and their derivatives). With this convention, $\partial_\mu\mathcal{L}$ would be zero, even though of course $\mathcal{L}$ changes as a function of position, because the fields do change. On a larger scale, the derivation of a conserved energy-momentum tensor from the Lagrange-density or the action is an example of a Lie-derivative.

E.10 *Maxwell's equations under discrete symmetries, revisited*

The behaviour of the Maxwell-equations under the three discrete symmetries charge conjugation $\mathcal{C}$, parity inversion $\mathcal{P}$ and time reversal $\mathcal{P}$ was already the subject of Sect. A.7, but can be extended to deal with covariant objects like $G^{\mu\nu}$, $\tilde{F}^{\mu\nu}$ or ∂_μ in a straightforward way. As before, we will treat the general case with electric charges $\jmath^\mu$ as well as magnetic charges $\imath^\mu$:

$$\partial_\mu G^{\mu\nu} = +\frac{4\pi}{c}\jmath^\nu \quad \text{and} \quad \partial_\mu \tilde{F}^{\mu\nu} = -\frac{4\pi}{c}\imath^\nu \tag{E.446}$$

In both cases the antisymmetry of the field tensors $G^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ makes sure that the currents are conserved, i.e. $\partial_\nu \jmath^\nu = 0$ and $\partial_\nu \imath^\nu = 0$.

∂_μ, combining spatial and temporal derivatives, transforms sensibly only under the combined $\mathcal{PT}$-operation: Clearly, $\mathcal{PT}x^\mu = -x^\mu$ and in consequence, $\mathcal{PT}\partial_\mu = -\partial_\mu$. The electric 4-current $\jmath^\mu$ transforms under $\mathcal{PT}$ like a velocity, $\mathcal{PT}\jmath^\mu = \jmath^\mu$, and under $\mathcal{C}$ as $\mathcal{C}\jmath^\mu = -\jmath^\mu$, and therefore $\mathcal{CPT}\jmath^\mu = -\jmath^\mu$ under the full $\mathcal{CPT}$ transform. Magnetic charges, however are pseudoscalar such that $\mathcal{PT}\imath^\mu = -\imath^\mu$, but in fact the additional minus sign does not matter when considering the continuities $\partial_\mu \jmath^\mu = 0$ and $\partial_\mu \imath^\mu = 0$.

Please note that one can only invoke arguments that relate $G^{\mu\nu}$ to the potential A_μ if there are no magnetic charges and duality is broken. It will be sufficient to consider the Faraday tensor $F_{\mu\nu}$ as its properties are identical to $G^{\mu\nu}$ because the two are related in a linear way by a mere prefactor. If there are only electric charges, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ suggests that $\mathcal{PT}F_{\mu\nu} = -F_{\mu\nu}$, using the wave equation $\Box A_\mu = 4\pi/c\ \eta_{\mu\nu}\jmath^\nu$, such that A_μ inherits its properties from $\jmath^\mu$, in summary $\mathcal{PT}A_\mu = +A_\mu$. This is consistent with the field equation $\partial_\mu G^{\mu\nu} = 4\pi/c\ \jmath^\mu$, as the minus signs brought in by ∂_μ and $G^{\mu\nu}$ cancel. Similarly, $\mathcal{PT}\tilde{F}^{\mu\nu} = +\tilde{F}^{\mu\nu}$ to reflect the plus-sign in $\mathcal{PT}\imath^\mu = +\imath^\mu$.

E.11 *Links to particle physics*

E.11.1 Axions and pseudoscalar particles

There is a second quadratic field invariant, $F_{\mu\nu}\tilde{F}^{\mu\nu} \propto E_i B^i$, which is pseudo-scalar: despite being "just" a number, it changes its sign under application of parity-transforms $\mathcal{P}$ and time reversal $\mathcal{T}$. This is the reason why we disregarded this particular term, despite being quadratic, as a contender for the Lagrange density $\mathcal{L}$ for electrodynamics. But multiplying with a field θ which itself is pseudoscalar, would amend this problem:

$$\mathcal{L} = \frac{\eta^{\alpha\mu}\eta^{\beta\nu}}{4} F_{\alpha\beta}F_{\mu\nu} + \alpha\theta F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{4\pi}{c} A_\mu \jmath^\mu + \frac{\eta^{\mu\nu}}{2}\partial_\mu\theta\,\partial_\nu\theta - V(\theta) \tag{E.447}$$

with a coupling strength α. This axion field θ needs its own dynamics and interacts with itself through the potential $V(\theta)$. Looking at the Taylor-expansion of $V(\theta)$ one can only admit even powers

$$V(\theta) = \sum_{n=0} \frac{\alpha_{2n}}{(2n)!}\theta^{2n} \tag{E.448}$$

as only those are invariant under parity transform: Essentially, this is a very strong restriction on the form of the potential for self-interaction of the axion field: it is necessarily an even function. Please note that a mass term of the type

$$V(\theta) = \frac{m^2}{2}\theta^2 \tag{E.449}$$

would be naturally contained in the interaction potential $V(\theta)$ even in the restriction to parity positive terms, by setting $\alpha_2 = m^2$ for $n = 1$.

Variation of the Lagrange-density with respect to A_μ yields an extension to the Maxwell field-equation, and the variation with respect to θ a corresponding equation of motion for θ, which is coupled to $F_{\mu\nu}$, i.e. a modified field equation

$$\eta^{\alpha\mu}\eta^{\nu\beta}\partial_\alpha F_{\mu\beta} = \frac{4\pi}{c}\jmath^\nu + \alpha\partial_\mu\left(\theta\tilde{F}^{\mu\nu}\right) = \frac{4\pi}{c}\jmath^\nu + \alpha\partial_\mu\theta\cdot\tilde{F}^{\mu\nu} \tag{E.450}$$

because $\partial_\mu\tilde{F}^{\mu\nu} = 0$, if duality is properly broken, and alongside a dynamical equation for θ,

$$\Box\theta = \alpha F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{dV}{d\theta} \tag{E.451}$$

Therefore, θ obeys a wave equation that is coupled to the electromagnetic field and driven by the potential gradient $-dV/d\theta$. Experiments with axions are always great and fascinating, for instance light through wall-type experiments. There, one tries to take a very strong photon source, such as a laser beam, and convert the photons by means of the $\theta F_{\mu\nu}\tilde{F}^{\mu\nu}$-term to axions. Clearly, $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is zero for a plane electromagnetic wave, so one provides an additional magnetic field to make the scalar product between the laser's electric field E_i and the external magnetic field B^i nonzero, enabling the conversion. The experimental setup continues then to block the laser beam by a wall and invert the conversion behind the wall, hopefully recoving photons from the axion field by supplying again a strong magnetic field.

E.11.2 Massive fields, Proca-terms and the Higgs-mechanism

For a scalar field φ it is rather straightforward to make it massive. In fact, it suffices to add a term that is quadratic in the field amplitude φ to the Lagrange-density. Then,

$$\mathcal{L}(\varphi, \partial_\mu\varphi) = \frac{\eta^{\alpha\beta}}{2}\partial_\alpha\varphi\partial_\beta\varphi - \frac{m^2}{2}\varphi^2. \tag{E.452}$$

Substitution into the Euler-Lagrange equation gives the equation of motion, which now reads

$$\Box\varphi = m^2\varphi \tag{E.453}$$

and a plane-wave ansatz of the type $\varphi \propto \exp(\pm \mathrm{i} k_\alpha x^\alpha)$ would yield as a dispersion relation

$$\eta^{\mu\nu}k_\mu k_\mu = \left(\frac{\omega}{c}\right)^2 - \gamma^{ij}k_i k_j = m^2 \quad \text{such that} \quad \omega = \pm c\sqrt{k^2 + m^2} \tag{E.454}$$

The wave number k_μ has clearly a time-like normalisation, $\eta^{\mu\nu}k_\mu k_\nu = m^2 > 0$, such that the propagation takes place inside the future light cone, as expected from a massive object. In addition, the group- and phase velocities are

$$\upsilon_\mathrm{gr} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c\frac{k}{\sqrt{k^2 + m^2}} < c \quad \text{and} \quad \upsilon_\mathrm{ph} = \frac{\omega}{k} = c\frac{\sqrt{k^2 + m^2}}{k} > c \tag{E.455}$$

because $\sqrt{k^2 + m^2} > k$, but their geometric mean is exactly

$$\upsilon_\mathrm{ph} \times \upsilon_\mathrm{gr} = c^2 \tag{E.456}$$

i.e. the phase velocity is superluminal, but the group velocity which is associated to the propagation speed of wave packets representing massive particles, remains subluminal. This is nicely illustrated by Fig. 25, where both velocities reach the same limiting value of c for $k \to \infty$, i.e. for $k \gg m$, as the mass becomes less and less relevant in that limit.

Motivated by this example one could think of a modified Lagrange density for the Maxwell field of the form

$$\mathcal{L} = \frac{\eta^{\alpha\mu}\eta^{\beta\nu}}{4}\mathrm{F}_{\alpha\beta}\mathrm{F}_{\mu\nu} + \frac{m^2}{2}\eta^{\alpha\mu}\mathrm{A}_\alpha\mathrm{A}_\mu \tag{E.457}$$

with a so-called Proca-term $\eta^{\alpha\mu}\mathrm{A}_\alpha\mathrm{A}_\mu$. Variation with the corresponding Euler-Lagrange equation would yield a seemingly sensible result, as

$$\eta^{\alpha\mu}\partial_\alpha\mathrm{F}_{\mu\nu} = \Box\mathrm{A}_\nu = m^2\mathrm{A}_\nu \tag{E.458}$$

in Lorenz-gauge, where the same plane-wave ansatz $\exp(\pm\mathrm{i}k_\alpha x^\alpha)$ would give a time-like normalisation $\eta^{\alpha\mu}k_\alpha k_\mu = m^2 > 0$ that corresponds to subluminal motion inside the light cone. But there is a fundamental problem already present in the Lagrange density: It is not gauge-invariant,

$$\eta^{\alpha\mu}\mathrm{A}_\alpha\mathrm{A}_\mu \to \eta^{\alpha\mu}\mathrm{A}_\alpha\mathrm{A}_\mu + 2\eta^{\alpha\mu}\partial_\alpha\chi\,\mathrm{A}_\mu + \eta^{\alpha\mu}\partial_\alpha\chi\,\partial_\mu\chi, \tag{E.459}$$

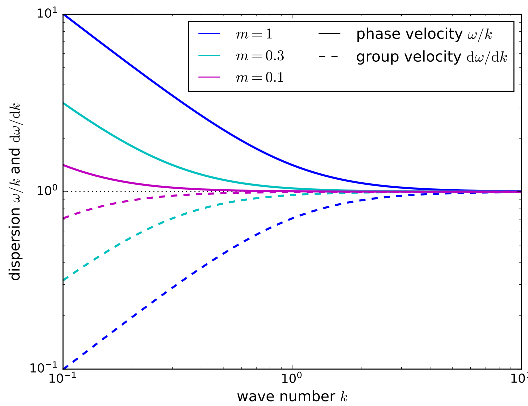


Figure 25: Dispersion relation, i.e. group and phase velocity as a function of wave number, for different masses.

so the choice of a suitable gauge is not possible. In fact, the problem of generating masses dynamically in a gauge-invariant way is solved only by the Higgs-mechanism for field theories and misses yet a complete solution for massive gravity. Electrodynamics as a theory without masses is backed up by stringent experimental upper bounds on the photon mass.

One should add, though, that Lorenz-gauge is still a very sensible choice for cases with a non-zero Proca-mass. Clearly, constructing the action S from the Lagrange-density eqn. (E.457) includes the additional terms

$$S = \int_V \mathrm{d}^4x \left(2\eta^{\alpha\mu}\partial_\alpha\chi A_\mu + \eta^{\alpha\mu}\partial_\alpha\chi\partial_\mu\chi\right) = -\int_V \mathrm{d}^4x \left(2\chi\underbrace{\eta^{\alpha\mu}\partial_\alpha A_\mu}_{=0} + \chi\underbrace{\eta^{\alpha\mu}\partial_\alpha\partial_\mu\chi}_{=\Box\chi}\right) \quad \text{(E.460)}$$

after integration by parts: In fact, Lorenz-gauge then makes the first term disappear and forces the gauge field χ to obey a wave-equation $\Box\chi = 0$.

E.11.3 Modifications of the Coulomb-potential

Scalar fields φ, even in the case of linear field equations, show an interesting phenomenology on large scales: Starting from the most general Lagrange-density $\mathcal{L}(\varphi, \partial_\mu\varphi)$ including all terms up to φ^2 would ensure a linear field equation, as in the variation process the powers get reduced by one:

$$\mathcal{L}(\varphi, \partial_\mu\varphi) = \frac{\gamma^{\mu\nu}}{2}\partial_\mu\varphi\,\partial_\nu\varphi - \frac{m^2}{2}\varphi^2 - 4\pi\rho\varphi + \lambda\varphi, \quad \text{(E.461)}$$

leading by variation to the field equation

$$(\Box + m^2)\varphi = 4\pi\rho + \lambda \quad \text{(E.462)}$$

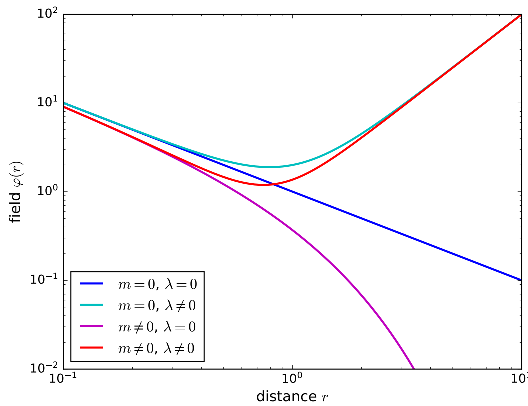


Figure 26: Field amplitude $\varphi(r)$ for the most general linear scalar field theory.

where one admits a source term ρ and an inhomogeneity λ, which would be present even in a charge free space and which would, in a gravitational theory, correspond to the gravitational constant. Focusing on a static, spherically symmetric situation for a point charge one recovers from the field equation

$$(\Delta - m^2)\varphi = -4\pi\rho - \lambda \quad \text{with} \quad \Delta\varphi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\varphi\right), \tag{E.463}$$

depending on the choice of the two parameters, the classical Coulomb-potential

$$\varphi(r) = \frac{1}{r} \tag{E.464}$$

for $m = 0 = \lambda$. Admitting a nonzero mass leads to the Yukawa-potential

$$\varphi(r) = \frac{\exp(-mr)}{r} \tag{E.465}$$

for $m \neq 0 = \lambda$, where the field amplitude φ is suppressed at large r. The full theory implies

$$\varphi(r) = \frac{\exp(-mr)}{r} + \lambda r^2 \tag{E.466}$$

for $m \neq 0 \neq \lambda$, with modifications large scales, while λ alone gives rise to

$$\varphi(r) = \frac{1}{r} + \lambda r^2 \tag{E.467}$$

for $m = 0 \neq \lambda$, which would, up to a sign, be the gravitational potential of a point mass in the classical limit including a cosmological constant. Common to the results are the definition of two additional length scales $1/m$ and $1/\sqrt{\lambda}$, which modify the otherwise scale-free Coulomb-solution. Fig. 26 summarises these modifications.

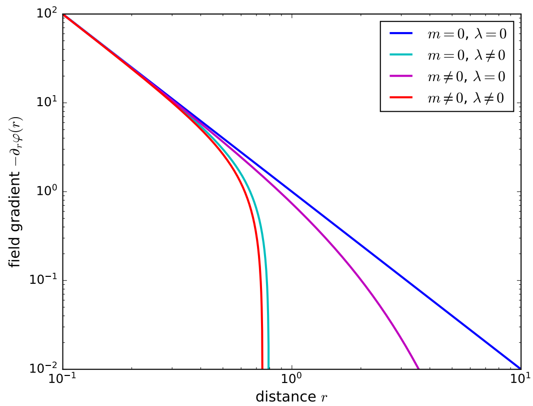


Figure 27: Field gradients $-\mathrm{d}\varphi/\mathrm{d}r$ *for the most general linear scalar field theory.*

The gradient $-\partial\varphi/\partial r$ would, if φ is interpreted as a potential, accelerate a test charge. The acceleration as a function of r is shown in Fig. 27, illustrating how on small scales $r \ll 1/m$ and $r \ll 1/\sqrt{\lambda}$, the unaffected Coulomb-potential is recovered, while there are modifications on large scales $r \gg 1/m$ and $r \gg 1/\sqrt{\lambda}$. It should be noted that the generalised inhomogeneity λ is not admissible in a non-scalar theory like electrodynamics, as a term linear in the 4-potential λA_μ is clearly non-scalar.

E.12 *Conformal invariance of the Maxwell-theory*

Apart from Lorentz- and gauge-invariance, and the spacetime shift symmetries of the Lagrange-density of Maxwell-electrodynamics there is, at least for vacuum-solutions, a weird scale-symmetry. Applying a rescaling of the spacetime coordinates

$$x^\alpha \rightarrow \lambda x^\alpha \quad \text{and consequently,} \quad \partial_\alpha \rightarrow \frac{1}{\lambda}\partial_\alpha. \tag{E.468}$$

The fields obey homogeneous wave equations in vacuum,

$$\Box F_{\mu\nu} = 0 \quad \text{and} \quad \Box G^{\mu\nu} = 0, \tag{E.469}$$

where in fact the λ^{-2} factor generated in $\Box \rightarrow \Box/\lambda^2$ drops out because of the vanishing right hand side of the two equations. This is an example of conformal symmetry. It is broken because the charge density ρ changes under the scaling $\propto \lambda^{-3}$ instead of $\propto \lambda{-}2$ as the differential operators.

E.13 *Gauge-invariance as a geometric concept*

The relationship between the fields and the derivatives in a relativistic notation are summarised by this diagram: The potential A_ν has an antisymmetric derivative $\tilde{F}^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}\partial_\mu A_\nu$, and this dual is divergence-free in fulfilment of the Bianchi-identity: $\partial_\alpha\tilde{F}^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}\partial_\alpha\partial_\mu A_\nu$, with an exchange symmetry in the index pair (α,μ) which makes the expression disappear in conjunction with the antisymmetry of the Levi-

Civita symbol: This is quite important because of two reasons: Not only does one recover the homogeneous Maxwell-equations, but it is clear that the potential A_μ is incompatible with a hypothetical nonzero magnetic source $\imath^\beta$.

Converting $\tilde{F}^{\alpha\beta}$ into $F_{\gamma\delta}$ and bringing in the constitutive relation yields the field tensor $G^{\gamma\delta}$. The divergence $\partial_\gamma G^{\gamma\delta}$ is the source $\jmath^\delta$, as an expression of the field equation. And finally, charge conservation in the sense of $\partial_\delta \jmath^\delta = 0$ is ensured by $\partial_\gamma\partial_\delta G^{\gamma\delta}$, again with a contraction of a symmetric with an antisymmetric tensor.

The gauge function χ changes A_μ, but leaves $F_{\mu\nu}$ invariant: This is accomplished by the derivative $\epsilon^{\alpha\beta\nu\mu}\partial_\nu\partial_\mu\chi = 0$, as the two derivatives interchange, $\partial_\mu\partial_\nu = \partial_\nu\partial_\mu$, but $\epsilon^{\alpha\beta\nu\mu}$ is antisymmetric in the index pair (μ, ν).

$$
\begin{array}{ccccccc}
 & & & & F_{\gamma\delta}, G^{\gamma\delta} & \xrightarrow{\partial_\gamma} & \jmath^\delta & \text{Maxwell} \\
 & & & & \uparrow \epsilon_{\alpha\beta\gamma\delta} & & & \\
 & & A_\nu & \xrightarrow{\epsilon^{\alpha\beta\mu\nu}\partial_\mu} & \tilde{F}^{\alpha\beta} & \xrightarrow{\partial_\alpha} & 0 & \text{Bianchi} \\
 & & \uparrow + & & \uparrow + & & & \\
\chi & \xrightarrow{\partial_\nu} & \partial_\nu\chi & \xrightarrow{\epsilon^{\alpha\beta\mu\nu}\partial_\mu} & 0 & & & \text{gauging}
\end{array}
\tag{E.470}
$$

Finding a gauge function χ for a given gauge condition, usually a derivative property of the potential like a particular value for $\eta^{\mu\nu}\partial_\mu A_\nu$ as in the Lorenz gauge requires the solution of a wave equation: Substitution $A_\mu \rightarrow A_\mu + \partial_\mu\chi$ into the gauge condition leads to $\eta^{\mu\nu}\partial_\mu\partial_\nu\chi = -b$ with $b = \eta^{\mu\nu}\partial_\mu A_\nu$. Wave-equations of this type are readily solvable by means of the retarded Green-functions.

$$
\begin{array}{cccccc}
 & & A_\mu & \xrightarrow{\eta^{\mu\nu}\partial_\nu} & b & \text{condition} \\
 & & \downarrow - & & \downarrow - & \\
\chi & \xrightarrow{\partial_\mu} & \partial_\mu\chi & \xrightarrow{\eta^{\mu\nu}\partial_\nu} & \Box\chi & \text{gauging}
\end{array}
\tag{E.471}
$$

The same diagram with identical arguments can be more concisely expressed in the language of differential forms: Starting from the 4-potential A_μ as a one-form A, application of the exterior derivative d leads to the two-form F, corresponding to the field tensor $F_{\mu\nu}$. The co-differential δ, which can be expressed as $\star d \star$ with the Hodge-star operator $\star$, leads to the source $\jmath$, again a one-form. The Hodge-dual of the field two-form F would be $\star F$, whose co-differential $\delta\star F = \star d \star\star F = \star dF = \star ddA = 0$, recovering the Bianchi-identity. The gauge field χ has an exterior derivative $d\chi$, which can be added to the one-form A without changing the observable fields contained in the two-form F, as $dA \rightarrow d(A + d\chi) = dA + dd\chi = dA$. On the other hand, $F = dA$ is only sensible if $\delta\star F = 0$ physically, i.e. that the magnetic charges are non-existent: The existence of a potential A requires broken duality.

$$\begin{array}{cccccccc} & & & & \star\mathrm{F} & \xrightarrow{\delta} & 0 & \text{Bianchi} \\ & & & & \uparrow{\scriptstyle\star} & & & \\ & & \mathrm{A} & \xrightarrow{\mathrm{d}} & \mathrm{F},\mathrm{G} & \xrightarrow{\delta} & \jmath & \text{Maxwell} \\ & & \uparrow{\scriptstyle +} & & \uparrow{\scriptstyle +} & & & \\ \chi & \xrightarrow{\mathrm{d}} & \mathrm{d}\chi & \xrightarrow{\mathrm{d}} & 0 & & & \text{gauging} \end{array} \tag{E.472}$$

The construction of a (scalar) gauge function χ for ensuring e.g. Lorenz-gauge $\delta\mathrm{A} = 0$ implies that $\mathrm{d}\chi$, now a one-form, is added to A and leads to $\delta\mathrm{d}\chi = -b$, with a source $b = \delta\mathrm{A}$ after substitution into the gauge condition. $\delta\mathrm{d}\chi$, however is the Laplace-de Rham-operator, which for our case of a Lorentzian metric background is the d'Alembert-operator $\Box$, up to a symmetrisation.

$$\begin{array}{cccccc} & & \mathrm{A} & \xrightarrow{\delta} & b & \text{condition} \\ & & \downarrow{\scriptstyle -} & & \downarrow{\scriptstyle -} & \\ \chi & \xrightarrow{\mathrm{d}} & \mathrm{d}\chi & \xrightarrow{\delta} & \Box\chi & \text{gauging} \end{array} \tag{E.473}$$

E.14 *Motion of particles through spacetime*

E.14.1 Fermat's or Hamilton's principle?

The relativistic expression for the arc length s through spacetime, as mapped out by proper time, can be amended by a second term, $q\mathrm{A}_\mu \mathrm{d}x^\mu$ which should incorporate the accelerating effects of electric and magnetic fields on a test particle with charge q:

$$s = \int_\mathrm{A}^\mathrm{B} \mathrm{d}\tau\, mc\sqrt{\eta_{\mu\nu}u^\mu u^\nu} + q\mathrm{A}_\mu \mathrm{d}x^\mu \quad \rightarrow \quad \mathcal{L}(x^\mu, u^\mu) = mc\sqrt{\eta_{\mu\nu}u^\mu u^\nu} + q\mathrm{A}_\mu u^\mu, \tag{E.474}$$

where in isolating the Lagrange function one rewrites $\mathrm{d}x^\mu = u^\mu \mathrm{d}\tau$, from the definition $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$. Variation of the arc-length, now a function of both u^μ and x^μ (through the coordinate dependence of A_μ) is achieved with the Euler-Lagrange equation,

⚠ The term $\mathrm{A}_\mu u^\mu$ emphasises how natural velocity-dependent forces in relativity are!

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial u^\alpha} = \frac{\partial\mathcal{L}}{\partial x^\alpha}. \tag{E.475}$$

The expression (E.474) for the relativistic arc length is remarkable, as it combines the metric distance in the first term with a second distance measure $\mathrm{A}_\mu \mathrm{d}x^\mu$ mediated by the vector potential, called ✈ Finsler geometry.

An intuitive (but gauge-dependent) picture might be, that different paths through spacetime have the particle change its proper time according to the magnitude and direction of A_μ relative to its velocity u^μ, like a tail- or headwind that changes travel time. The necessary derivatives are

$$\frac{\partial\mathcal{L}}{\partial x^\alpha} = q u^\mu \partial_\alpha A_\mu \tag{E.476}$$

and

$$\frac{\partial\mathcal{L}}{\partial u^\alpha} = m\eta_{\mu\alpha} u^\mu + qA_\alpha \quad\rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial u^\alpha} = m\eta_{\mu\alpha}\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + qu^\mu\partial_\mu A_\alpha \tag{E.477}$$

where the last term appears in the time derivative through the coordinate dependence of A_α:

$$\frac{\mathrm{d}A_\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\partial A_\alpha}{\partial x^\mu} = u^\mu\,\partial_\mu A_\alpha. \tag{E.478}$$

Collecting all results yields

$$m\eta_{\mu\alpha}\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = q\left(\partial_\alpha A_\mu - \partial_\mu A_\alpha\right)u^\mu = qF_{\alpha\mu}u^\mu \tag{E.479}$$

by identifying the Faraday-tensor in the last step: Finally, we recover the Lorentz equation of motion, and the appearance of $F_{\mu\nu}$ makes sure that the acceleration does not depend on gauge. Multiplying both sides of the equation with u^α leads to an interesting result:

$$\eta_{\mu\alpha}u^\alpha\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\eta_{\mu\alpha}u^\alpha u^\mu\right) = qF_{\alpha\mu}u^\alpha u^\mu = 0, \tag{E.480}$$

where the last term is necessarily zero as the contraction between the symmetric tensor $u^\alpha u^\mu$ and the antisymmetric $F_{\alpha\mu}$. This safeguards the norm $\eta_{\mu\alpha}u^\alpha u^\mu = c^2$ from any changes, and keeps the particle from being accelerated to superluminal velocities outside the light cone.

While the equation of motion is perfectly gauge-invariant (and Lorentz-covariant), the gauge-invariance of the Lagrange-function requires additional arguments: Performing a gauge transform $A_\mu \rightarrow A_\mu + \partial_\mu\chi$ with a gauge function χ changes the relativistic arc length according to

$$s = \int_A^B \mathrm{d}\tau\left(mc\sqrt{\eta_{\mu\nu}u^\mu u^\nu} + qA_\mu u^\mu\right) \quad\rightarrow\quad s + q\int_A^B \mathrm{d}\tau\,\partial_\mu\chi\,u^\mu. \tag{E.481}$$

This new term can be rewritten, by falling back onto the form how it was introduced,

$$\int_A^B \mathrm{d}\tau\,\partial_\mu\chi\,u^\mu = \int_A^B \mathrm{d}x^\mu\partial_\mu\chi = \int_A^B \mathrm{d}\chi = (\chi_B - \chi_A), \tag{E.482}$$

using $\mathrm{d}\chi = \partial_\mu\chi\mathrm{d}x^\mu$. In summary, there is a constant, additive term that becomes irrelevant for the variation for obtaining the trajectory.

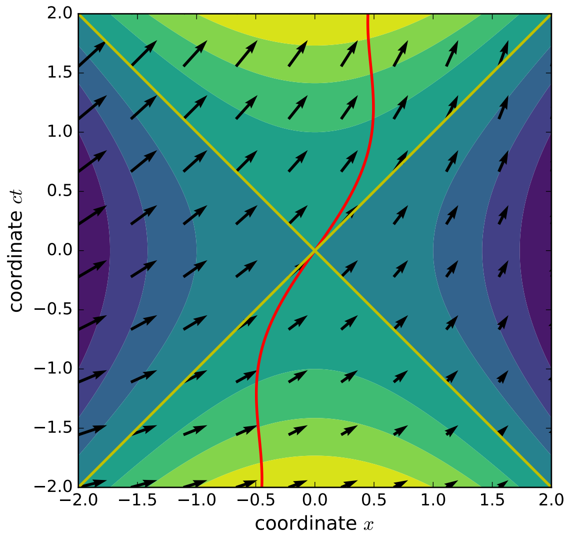


Figure 28: Trajectory through spacetime, with the metric contribution $\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ *to the line element* $\mathrm{d}s^2$ *in the background shading and the Finsler contribution* $A_\mu\mathrm{d}x^\mu$ *generated by the potential* A_μ *as arrows.*

An impression on the contributions to the line element $\mathrm{d}s^2$ given by the metric $\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ and the Finsler-term $A_\mu\mathrm{d}x^\mu$ is given in Fig. 28.

E.14.2 Relativistic horizons

We can probe the limits of special relativity by looking at accelerated, non-inertial motion through spacetime. Starting from the coordinates x^μ we already defined the 4-velocity u^μ,

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} ct \\ x \end{pmatrix} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} ct \\ x \end{pmatrix} = \gamma\begin{pmatrix} c \\ \upsilon \end{pmatrix} \tag{E.483}$$

with $\upsilon = \dot{x}$ and $\gamma = \mathrm{d}t/\mathrm{d}\tau$. Repeating this argument one computes the 4-acceleration a^μ as

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}\gamma\begin{pmatrix} c \\ \upsilon \end{pmatrix} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t}\gamma\begin{pmatrix} c \\ \upsilon \end{pmatrix} = \frac{\upsilon a}{c^2}\gamma^4\begin{pmatrix} c \\ \upsilon \end{pmatrix} + \gamma^2\begin{pmatrix} 0 \\ a \end{pmatrix} \tag{E.484}$$

with $a = \dot{\upsilon} = \ddot{x}$, and the derivative $\mathrm{d}\gamma/\mathrm{d}t = \gamma^3\upsilon a/c^2$. This system of equations can be integrated numerically for e.g. an assumed constant acceleration a, giving a parametric solution $(ct(\tau), x(\tau))$. The resulting trajectories in $x^\mu(\tau)$ are shown in Fig. 29, where the accelerated trajectory evades light signals that are emitted at $x = 0$ later than $ct \geq 4$, which is impossible for inertial motion. Effectively, evading light signals means that there is a relativistic horizon between the emitter of light signals and the accelerated particle.

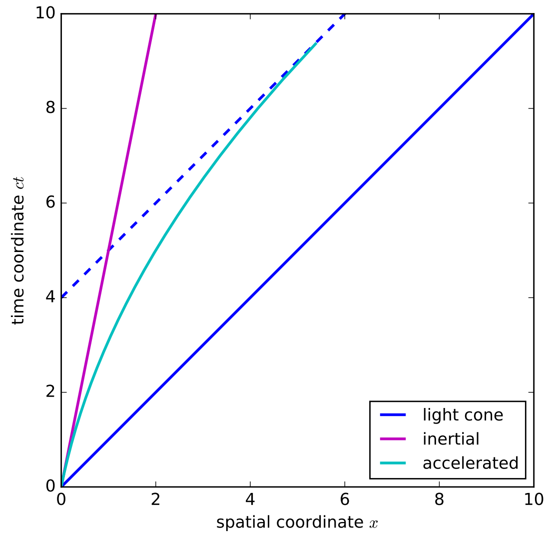


Figure 29: Paths through spacetime at constant velocity, and in comparison a path with constant acceleration, with the emergence of a relativistic horizon.

The 4-acceleration a^μ is always perpendicular to the 4-velocity u^μ,

$$\eta_{\mu\nu}u^\mu a^\nu = 0, \tag{E.485}$$

as a direct computation with the above expression shows. This has in fact dramatic consequences, as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\eta_{\mu\nu}u^\mu u^\nu\right) = \eta_{\mu\nu}\left(\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}u^\nu + u^\mu\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}\right) = 2\eta_{\mu\nu}u^\mu a^\nu = 0, \tag{E.486}$$

implying that the (timelike) norm $\eta_{\mu\nu}u^\mu u^\nu = c^2 > 0$ of u^μ is conserved. At this point it is worth mentioning that many texts attribute the impossibility of accelerating a massive object past c to the relativistic mass increase, which is really superfluous as a concept as it is completely covered by the geometric, kinematical structure of spacetime. Proper acceleration is defined in terms of proper time τ, which is dilated relative to the coordinate time t by the Lorentz-factor γ. A faster-moving system reacts to an accelerating force as if it had more inertia and therefore a higher mass, but it is really the conversion between proper time and coordinate time that brings in the Lorentz-factor, and one does not need to invoke a new relativistic effect on mass, and surely the number of atoms inside an object would be unchanged under Lorentz transforms!

"The concept of "relativistic mass" is subject to misunderstanding. That's why we don't use it. First, it applies the name mass – belonging to the magnitude of a 4-vector – to a very different concept, the time component of a 4-vector. Second, it makes increase of energy of an object with velocity or momentum appear to be connected with some change in internal structure of the object. In reality, the increase of energy with velocity originates not in the object but in the geometric properties of spacetime itself.", E. F. Taylor and J. A. Wheeler, Spacetime Physics

E.14.3 Tachyons and tardyons

Tachyons are hypothetical, superluminally moving particles with 4-velocities u^μ outside the light cone, $\eta_{\mu\nu}u^\mu u^\nu = -c^2 < 0$. On the other side, tardyons are

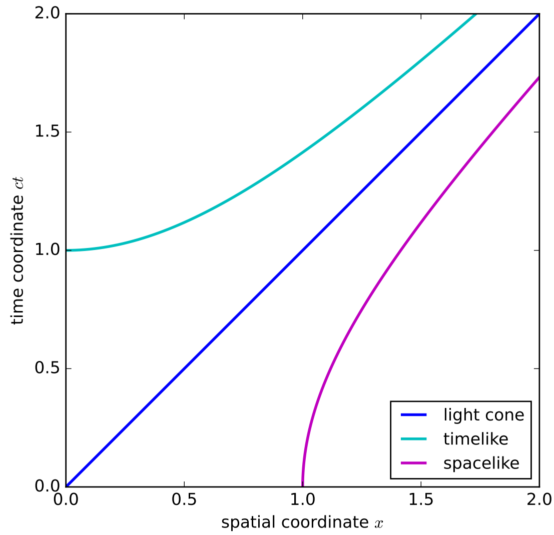


Figure 30: Curves of constant Minkowski-norm $\eta_{\mu\nu}x^\mu x^\nu = \pm 1$, or equivalently, curves traced out by the endpoint of a timelike and spacelike unit vector under Lorentz-transforms.

conventional, massive particles with subluminal velocities inside the light cone, $\eta_{\mu\nu}u^\mu u^\nu = +c^2 > 0$. Naturally, these norms are conserved under Lorentz-transforms, as illustrated by Fig. 30, where the hyperbolic curves traced out by the unit vectors along the x- and ct-axes never leave their associated timelike or spacelike quadrants. For a timelike vector this would be,

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh\psi \\ \sinh\psi \end{pmatrix} \tag{E.487}$$

and for a spacelike vector correspondingly,

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sinh\psi \\ \cosh\psi \end{pmatrix}. \tag{E.488}$$

For a particle moving on a spacelike trajectory one would write down a line element

$$c^2\mathrm{d}\tau^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = c^2\mathrm{d}t^2 - \gamma_{ij}\mathrm{d}x_i\mathrm{d}x_j = \left(c^2 - \gamma_{ij}\upsilon^i\upsilon^j\right)\mathrm{d}t^2 \tag{E.489}$$

with $\upsilon^i = \mathrm{d}x^i/\mathrm{d}t$. Negative norms would then imply that $\gamma_{ij}\upsilon^i\upsilon^j > c^2$, and hence that the magnitude of υ exceeds c. The velocity u^μ for such a particle would necessarily have the same negative norm, as one writes $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, and because $c^2\mathrm{d}\tau^2 = \eta_{\mu\nu}u^\mu u^\nu\mathrm{d}\tau^2$ has to have the same overall sign.

The relativistic dispersion relation $\mathcal{H}^2 = (cp)^2 + (mc^2)^2$ suggests the definition of a relativistic 4-momentum p_μ (as a linear form), whose norm is positive for tardyonic and negative for tachyonic particles, according to the location of the corresponding velocities in the respective quadrants in a spacetime diagram,

$$p_\mu = \begin{pmatrix} \mathcal{H}, & cp_i \end{pmatrix} \quad \text{with} \quad \eta^{\mu\nu} p_\mu p_\nu = \mathcal{H}^2 - c^2 \gamma^{ij} p_i p_j = \mathcal{H}^2 - (cp)^2 = \pm (mc^2)^2, \tag{E.490}$$

resulting in a funny shape of the dispersion relation,

$$\mathcal{H}(p) = \sqrt{(cp)^2 \pm (mc^2)^2}, \tag{E.491}$$

for the negative sign: This is in fact consistent with their superluminality, as p^2 is bounded from below by $(mc)^2$: Tachyons need to be faster than the speed of light, and if they brake down to approach the speed of light from above, they can only reach mc. In a weird sense, this is analogous to the non-vanishing energy associated with the rest mass for normal, tardyonic particles: While for them the energy is nonzero even for vanishing momenta, tachyons have a minimal momentum even at zero energies. To some degree of overinterpretation, tachyons have a minimal momentum mc whereas the tardyons have a minimal energy mc^2. Reexpressing the tachyonic dispersion relation in terms of wave number and angular frequency would be

$$\omega = \pm c\sqrt{k^2 - m^2} \tag{E.492}$$

Group and phase velocities for tachyons come out as

$$\upsilon_{\mathrm{gr}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{ck}{\sqrt{k^2 - m^2}} > c \quad \text{and} \quad \upsilon_{\mathrm{ph}} = \frac{\omega}{k} = \frac{c\sqrt{k^2 - m^2}}{k} < c, \tag{E.493}$$

exactly inverted compared to massive particles: The group velocity, associated with particle propagation, is always superluminal because $\sqrt{k^2 - m^2} < k$, and the phase velocity subluminal. Their geometric average, though, comes out as

$$\upsilon_{\mathrm{gr}} \times \upsilon_{\mathrm{ph}} = c^2. \tag{E.494}$$

Of course one should keep in mind that outside the light cone there is no causal ordering due to the relativity of simultaneity, so it would be problematic to have tachyons influence the causal world inside the light cone. To conclude, there is no place for tachyons in a Galilean world: In the formal limit of $c \to \infty$, the future light cone opens up: The timelike region increases while the spacelike region decreases, until all of spacetime reaches an absolute causal ordering according to the universal, Galilean time. And of course, every velocity is subluminal as $c \to \infty$.

X FOURIER-TRANSFORMS AND ORTHONORMAL SYSTEMS

X.1 *Scalar products and orthogonality*

The fundamental idea of Fourier-transforms is the question whether a function can be represented as a linear combination of a parameterised family of base functions which acts as a basis system, very much like the representation of a vector in terms of its basis. For this purpose, one needs to generalise the notion of a projection to functions, i.e. one needs to define a sensible scalar product. Scalar products in vector spaces over $\mathbb{R}$ have the properties

1. positive definiteness:

   $\langle u, u\rangle \geq 0$, and $\langle u, u\rangle = 0$ implies $u = 0$

2. bilinearity:

   $\langle u, v_1 + v_2\rangle = \langle u, v_1\rangle + \langle u, v_2\rangle$ as well as $\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle$, and

   $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$ as well as $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$

3. symmetry:

   $\langle u, v\rangle = \langle v, u\rangle$

whereas in vector spaces over $\mathbb{C}$ there are slight differences,

1. positive definiteness:

   $\langle u, u\rangle \geq 0$, and $\langle u, u\rangle = 0$ implies $u = 0$

2. sesquilinearity (instead of bilinearity):

   $\langle u, v_1 + v_2\rangle = \langle u, v_1\rangle + \langle u, v_2\rangle$ as well as $\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle$, and

   $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$ but $\langle \alpha u, v\rangle = \alpha^*\langle u, v\rangle$ with a complex conjugation

3. hermiticity (instead of symmetry):

   $\langle u, v\rangle = \langle v, u\rangle^*$

In analogy to the scalar product in $\mathbb{R}^n$ one can define a scalar product for $\mathbb{R}$-valued functions in the interval $[a, b]$,

$$\langle u, v\rangle = u_i v^i \quad \rightarrow \quad \langle u, v\rangle = \int_a^b \mathrm{d}x\, u(x)v(x) \tag{X.495}$$

and for complex scalar products in $\mathbb{C}^n$ and $\mathbb{C}$-valued functions

$$\langle u, v\rangle = u_i^* v^i \quad \rightarrow \quad \langle u, v\rangle = \int_a^b \mathrm{d}x\, u^*(x)v(x) \tag{X.496}$$

with a complex conjugation.

The notion of orthogonality

$$\langle u^i, u_j \rangle \propto \delta^i_j \tag{X.497}$$

generalises straightforwardly to a set of functions $u^{(i)}(x)$ indexed by i, where we denote functions as vectors with a basis $|u_i\rangle$ and the associated linear forms with a basis $\langle u^i|$, borrowing the bra-ket notation from quantum mechanics.

If such as set should be able to approximate a function $g(x)$ in a linear combination

$$g(x) = a^i |u_i(x)\rangle \tag{X.498}$$

needs to make sure that the quadratic error Δ_N

$$\langle a_i u^i(x) - g(x) | a^j u_j(x) - g(x) \rangle \tag{X.499}$$

between the function and its approximation over the interval $[a, b]$ becomes small, and ideally vanishes in the limit $\mathrm{N} \rightarrow \infty$. It is sensible to integrate up the quadratic difference because the linear combination can over- or underestimate $g(x)$: Δ_N is positive definite and vanishes in the case of a perfect approximation.

$$\Delta_\mathrm{N} = a_i^* a^j \langle u^i, u_j \rangle - a_i^* \langle u^i, g \rangle - a^j \langle g, u_j \rangle + \langle g, g \rangle \tag{X.500}$$

If the basis system of functions $|u_i(x)\rangle$ is chosen to be orthogonal,

$$\langle u^i, u_j \rangle = \int_a^b \mathrm{d}x\, u^{(i)}(x)^* u^{(j)}(x) = \delta^i_j \tag{X.501}$$

the double sum in the first term collapses to a single sum, such that

$$\Delta_\mathrm{N} = a_i^* a^i - a_i^* \langle u^i, g \rangle - a^i \langle g, u_i \rangle + \langle g, g \rangle \tag{X.502}$$

The squared error Δ_N can be minimised with respect to a^k and a_k^*, which are mutually independent (think of them as being complex numbers, clearly the real and imaginary part are independent)

$$\frac{\partial}{\partial a^k}\Delta_\mathrm{N} = \underbrace{\frac{\partial a_i^*}{\partial a^k}}_{=0} a_i + a_i^* \underbrace{\frac{\partial a^i}{\partial a^k}}_{=\delta^i_k} - \underbrace{\frac{\partial a_i^*}{\partial a^k}}_{=0} \langle u^i, g \rangle - \underbrace{\frac{\partial a^i}{\partial a^k}}_{=\delta^i_k} \langle g, u_i \rangle + \underbrace{\frac{\partial}{\partial a^k}\langle g, g \rangle}_{=0} \tag{X.503}$$

such that

$$\frac{\partial}{\partial a_k}\Delta_\mathrm{N} = a_k^* - \langle g, u^k \rangle = 0 \quad \rightarrow \quad a_k* = \langle g, u^k \rangle = \int_a^b \mathrm{d}x\, g(x)^* u^{(k)}(x) \tag{X.504}$$

Similarly, minimisation with respect to a_k^* yields

$$\frac{\partial}{\partial a_k^*}\Delta_\mathrm{N} = \underbrace{\frac{\partial a_i^*}{\partial a_k^*}}_{=\delta_i^k} a^i + a_i^* \underbrace{\frac{\partial a^i}{\partial a_k^*}}_{=0} - \underbrace{\frac{\partial a_i^*}{\partial a_k^*}}_{=\delta_i^k} \langle u^i, g\rangle - \underbrace{\frac{\partial a_i}{\partial a_k^*}}_{=0} \langle g, u_i\rangle + \underbrace{\frac{\partial}{\partial a_k^*}\langle g, g\rangle}_{=0} \tag{X.505}$$

implying

$$\frac{\partial}{\partial a_k^*}\Delta_\mathrm{N} = a^k - \langle u_k, g\rangle \quad\rightarrow\quad a_k = \langle u_k, g\rangle = \int_a^b \mathrm{d}x\, u_k(x)^* g(x) \tag{X.506}$$

which is the hermitean conjugate of eqn. (X.504): When determining the expansion coefficients a_k of a complex function $g(x)$, one directly obtains *both* the real and imaginary part of a_k from the projection integral, so $a_k^* = \langle g, u_k\rangle$ and $a^k = \langle u^k, g\rangle$ are equivalent. In the case of a real-valued function $g(x)$, both a_k^* and a^k coincide, which implies that the coefficients themselves are real-valued.

With the coefficents a^i and a_i^* derived by projection, the value of the squared error Δ_N at the minimum is given by

$$\Delta_\mathrm{N}^{(\mathrm{min})} = \langle g, g\rangle - a_i^* a^i \tag{X.507}$$

which ideally would tend towards zero as $\mathrm{N} \rightarrow \infty$,

$$\lim_{\mathrm{N}\to\infty} \Delta_\mathrm{N}^{(\mathrm{min})} = \lim_{\mathrm{N}\to\infty} \langle a_i u^i - g, a^i u_i - g\rangle = 0 \quad\rightarrow\quad \langle g, g\rangle = \lim_{\mathrm{N}\to\infty} a_i^* a^i \tag{X.508}$$

referred to as convergence in the quadratic mean, implying the Parseval-relation, which is tightly related to the completeness relation of the basis system: After all, not all basis systems are able to make sure that the minimised mean quadratic error tends to zero.

$$a_i^* a^i = \langle g, u^i\rangle \langle u_i, g\rangle = \int_a^b \mathrm{d}x\, g(x)^* u_i(x) \int_a^b \mathrm{d}x'\, u^i(x')^* g(x') \tag{X.509}$$

Changing the integration order leads to

$$a_i^* a^i = \int_a^b \mathrm{d}x\, g(x)^* \int_a^b \mathrm{d}x'\, g(x')\, u^i(x')^* u_i(x) \tag{X.510}$$

If the system of functions fulfils

$$u^i(x)^* u_i(x') = \delta_D(x - x') \tag{X.511}$$

then one can continue to write

$$a_i^* a^i = \int_a^b \mathrm{d}x\, g(x)^* \int_a^b \mathrm{d}x'\, g(x')\delta_D(x - x') = \int_a^b \mathrm{d}x\, g(x)^* g(x) = \langle g, g\rangle \tag{X.512}$$

and convergence is assured. This means, that the system of functions $|u^i(x)\rangle$ needs to be able to represent the Dirac δ_D-function. If that is the case, the system is complete for representing any function in the quadratic mean.

X.2 *Fourier-transforms*

Popular basis functions are plane waves because many differential equations in physics actually describe oscillations. In the finite interval $[-\pi, +\pi] \subset \mathbb{R}$, a discrete set of plane waves $u_n = \exp(\mathrm{i}nx)$ would be perfectly suited as a complete basis system, because

$$\sum_n^{\mathrm{N}} \exp(\mathrm{i}nx)\exp(-\mathrm{i}nx') = \sum_n^{\mathrm{N}} \exp(\mathrm{i}n(x - x')) =$$

$$\sum_n^{\mathrm{N}} \exp(\mathrm{i}(x - x'))^n = \frac{\exp(\mathrm{i}(x - x')(\mathrm{N}+1)) - 1}{\exp(\mathrm{i}(x - x')) - 1} \tag{X.513}$$

as a consequence of the limit formula for geometric series, which can be reformulated to yield

$$= \exp\left(\mathrm{i}\frac{\mathrm{N}}{2}(x - x')\right) \frac{\sin\left(\frac{\mathrm{N}+1}{2}(x - x')\right)}{\sin\left(\frac{1}{2}(x - x')\right)} \sim \delta_D(x - x') \tag{X.514}$$

as the exponential becomes 1 in the limit $x \to x'$, the $\sin(x)/x$-function indeed approximates the Dirac δ_D-function. To show that the value at $x = x'$ is actually proportional to $\mathrm{N} + 1$ requires the application of de l'Hôpital's rule for computing the limit $x \to x'$.

For the case of the infinite interval $(-\infty, +\infty)$ one can transition to a continuous set of basis functions. Introducing a wave vector $k = 2\pi/\mathrm{L}$ for a plane wave $\exp(2\pi\mathrm{i}x/\mathrm{L}) = \exp(\mathrm{i}kx)$ in the interval is likewise a complete basis system, and becomes continuous in the limit $\mathrm{L} \to 0$. In fact,

$$\int_{-\pi/\mathrm{L}}^{+\pi/\mathrm{L}} \frac{\mathrm{d}k}{2\pi} \exp(\mathrm{i}kx)\exp(\mathrm{i}kx')^* = \int_{-\pi/\mathrm{L}}^{+\pi/\mathrm{L}} \frac{\mathrm{d}k}{2\pi} \exp(\mathrm{i}k(x - x')) = \frac{1}{2\pi} \frac{\exp(\mathrm{i}k(x - x'))}{\mathrm{i}(x - x')}\Bigg|_{-\pi/\mathrm{L}}^{+\pi/\mathrm{L}} \tag{X.515}$$

and evaluating the integral yields

$$= \frac{1}{\pi} \frac{\sin(\pi(x - x')/\mathrm{L})}{\pi(x - x')/\mathrm{L}} \tag{X.516}$$

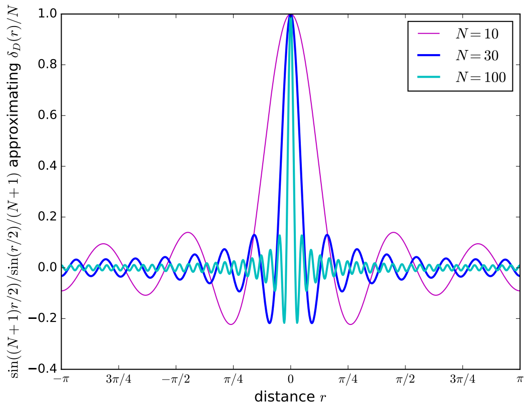


Figure 31: Eqn. X.516 as an approximation to the δ_D-function in the limit $N \to \infty$.

which in the limit $L \to 0$ behaves like the Dirac δ_D-function: The case of $x- \to x'$ can be sorted out by application of de l'Hôpital's rule, just as before in the discrete case.

In the continuum limit, the Fourier-transform $g(k)$ of a function $g(x)$ is given by

$$g(x) = \int \frac{\mathrm{d}k}{2\pi}\, g(k) \exp(+\mathrm{i}kx) \quad \leftrightarrow \quad g(k) = \int \mathrm{d}x\, g(x) \exp(-\mathrm{i}kx) \tag{X.517}$$

where you'll find in the literature any combination of distributing the factor 2π and choosing the sign in the wave $\exp(\pm\mathrm{i}kx)$. The two are really inverse, as

$$g(x) = \int \frac{\mathrm{d}k}{2\pi} \int \mathrm{d}x'\, g(x') \exp(\mathrm{i}k(x-x')) = \int \mathrm{d}x'\, g(x') \int \frac{\mathrm{d}k}{2\pi} \exp(\mathrm{i}k(x-x')) = \int \mathrm{d}x'\, g(x')\delta_D(x-x') = g(x) \tag{X.518}$$

illustrating the necessity of the 2π-factor. Generalising to more dimensions it becomes clear that the plane wave $\exp(\pm\mathrm{i}k_i r^i)$ factorises in Cartesian coordinates into $\exp(\pm\mathrm{i}k_x x)\exp(\pm\mathrm{i}k_y y)\exp(\pm\mathrm{i}k_z z)$, such that the Fourier-transform in n dimensions becomes a sequence of Fourier-transforms in 1 dimension:

$$g(\boldsymbol{r}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, g(\boldsymbol{k}) \exp(+\mathrm{i}k_i r^i) \quad \leftrightarrow \quad g(\boldsymbol{k}) = \int \mathrm{d}^3r\, g(\boldsymbol{r}) \exp(-\mathrm{i}k_i r^i) \tag{X.519}$$

Any further simplification is only possible if the function to be transformed itself factorises, too. The (scalar) product $\boldsymbol{k}\cdot\boldsymbol{r} = k_i r^i$ in index notation shows that k_i is in fact a linear form, which is foreshadowing quantum mechanics that sets $\hbar k_i = p_i$ with the momentum p_i.

X.3 *Convolutions with Fourier-transforms*

One of the primary applications of Fourier-transforms is to carry out convolutions $\varphi \otimes \psi$, as convolutions reduce to straightforward multiplications in Fourier-space.

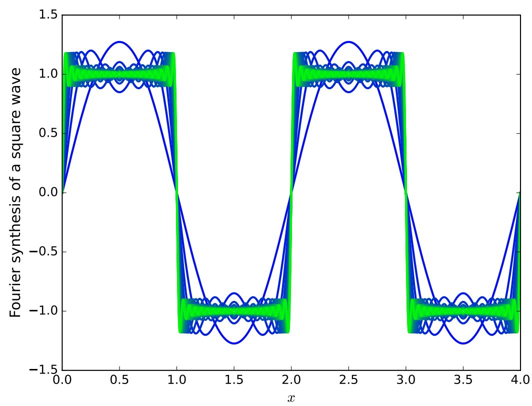


Figure 32: Square wave, assembled from the first 20 Fourier components.

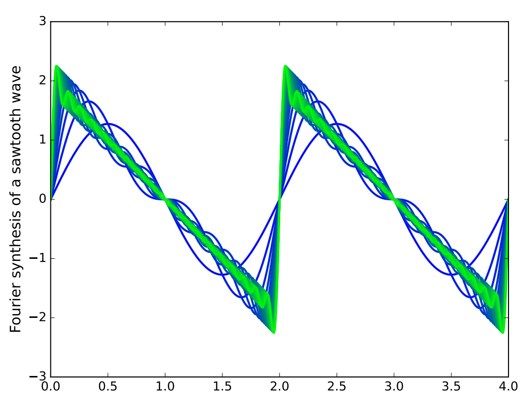


Figure 33: Sawtooth wave, assembled from the first 20 Fourier components.

Setting up a product $\varphi(\boldsymbol{k})\psi(\boldsymbol{k})$ between two Fourier-transformed functions $\varphi(\boldsymbol{k})$ and $\psi(\boldsymbol{k})$ and transforming back to configuration space yields

$$\varphi \otimes \psi(\boldsymbol{r}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, [\varphi(\boldsymbol{k})\psi(\boldsymbol{k})]\exp(+\mathrm{i}k_i r^i) \tag{X.520}$$

and substituting the forward-transformed fields gives

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \int \mathrm{dV}'\, \varphi(\boldsymbol{r}')\exp(-\mathrm{i}k_i r'^i) \int \mathrm{dV}''\, \psi(\boldsymbol{r}'')\exp(-\mathrm{i}k_j r''^j)\exp(+\mathrm{i}k_k r^k) \tag{X.521}$$

which, after reordering the integrations, is equivalent to

$$= \int \mathrm{dV}'\, \varphi(\boldsymbol{r}') \int \mathrm{dV}''\, \psi(\boldsymbol{r}'') \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \exp(+\mathrm{i}k_i \cdot [\boldsymbol{r} - \boldsymbol{r}' - \boldsymbol{r}'']^i) \tag{X.522}$$

The d^3k-integration gives the Dirac δ_D-function, which fixes $\boldsymbol{r}''$ to the value $\boldsymbol{r} - \boldsymbol{r}'$,

$$= \int \mathrm{dV}'\, \varphi(\boldsymbol{r}') \int \mathrm{dV}''\psi(\boldsymbol{r}'')\delta_\mathrm{D}(\boldsymbol{r} - \boldsymbol{r}' - \boldsymbol{r}'') = \int \mathrm{dV}'\, \varphi(\boldsymbol{r}')\psi(\boldsymbol{r} - \boldsymbol{r}') \tag{X.523}$$

i.e. a convolution, as advertised. Due to the perfect symmetry between Fourier-space and configuration space, the opposite is true as well: Convolutions in Fourier-space are products in configuration space.

X.4 *Green-functions with Fourier-transforms*

In the discussion of Poisson-type equations $\Delta\Phi = -4\pi\rho$ for solving potential problems we have seen that the potential Φ is given by a convolution of the charge distribution ρ with the Green-function G, which incidentally is $1/r$ for the Δ-operator in 3 dimensions:

$$\Phi(\boldsymbol{r}) = \int \mathrm{dV}'\, \mathrm{G}(\boldsymbol{r} - \boldsymbol{r}')\rho(\boldsymbol{r}') = \int \mathrm{dV}'\, \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{X.524}$$

This convolution needs to become a product in Fourier-space

$$\Phi(\boldsymbol{k}) = \mathrm{G}(\boldsymbol{k})\rho(\boldsymbol{k}) \quad \text{with} \quad \mathrm{G}(\boldsymbol{k}) = \frac{4\pi}{k^2} \tag{X.525}$$

To obtain the expression for the Green-function G in configuration space it suffices to transform $\mathrm{G}(k)$ back, where we make the replacement $\boldsymbol{r} - \boldsymbol{r}' \to \boldsymbol{r}'$, as the Green-function only depends on the relative distance:

$$\mathrm{G}(\boldsymbol{r}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \mathrm{G}(k)\exp(+\mathrm{i}k_i r^i) \tag{X.526}$$

As $\mathrm{G}(k) = 4\pi/k^2$ is spherically symmetric, it makes sense to carry out the integration in spherical coordinates: $\mathrm{d}^3k = k^2\mathrm{d}k\, \mathrm{d}\mu\, \mathrm{d}\varphi$ with $\mu = \cos\theta$ being the cosine of the polar angle θ:

$$G(\boldsymbol{r}) = \int_0^\infty \frac{k^2 \mathrm{d}k}{(2\pi)^3} \int_{-1}^{+1} \mathrm{d}\mu \int_0^{2\pi} \mathrm{d}\varphi \, \frac{4\pi}{k^2} \exp(+\mathrm{i}kr\mu) = \frac{4\pi}{(2\pi)^2} \int_0^\infty \mathrm{d}k \int_{-1}^{+1} \mathrm{d}\mu \, \exp(\mathrm{i}kr\mu) \quad \text{(X.527)}$$

because $k_i r^i = kr\cos\theta = kr\mu$, and because d$\varphi$-integration just yields 2π. Next, the dμ-integration can be carried out to yield

$$= \frac{1}{\pi} \int_0^\infty \mathrm{d}k \, \frac{\exp(+\mathrm{i}kr) - \exp(-\mathrm{i}kr)}{\mathrm{i}kr} = \frac{2}{\pi} \int_0^\infty \mathrm{d}k \, \frac{\sin(kr)}{kr} = \frac{2}{\pi} \int_0^\infty \mathrm{d}k \, j_0(kr) = \frac{1}{r} \quad \text{(X.528)}$$

because the integral over $j_0(x) = \sin(x)/x$ can be shown to be

$$\int_0^\infty \mathrm{d}x \, \frac{\sin(x)}{x} = \frac{\pi}{2} \quad \text{(X.529)}$$

after substitution $x = kr$, usually with the methods of complex calculus (see chapter Y), but there are more down-to-Earth methods: There is **no** direct integration method for this type of integral, but neat tricks exist!

$$\int_0^\infty \mathrm{d}x \, \frac{\sin x}{x} = \int_0^\infty \mathrm{d}x \, \sin(x) \underbrace{\int_0^\infty \mathrm{d}y \, \exp(-yx)}_{=1/x} = \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}x \, \sin(x)\exp(-yx) \quad \text{(X.530)}$$

after changing the order of integration. The resulting dx-integral can be solved by double integration by parts:

$$\int_0^\infty \mathrm{d}x \, \sin(x)\exp(-yx) = -\frac{1}{y}\sin(x)\exp(-yx)\Big|_0^\infty + \frac{1}{y}\int_0^\infty \mathrm{d}x \, \cos(x)\exp(-yx) \quad \text{(X.531)}$$

where the first term vanishes at both boundaries. Continuing with the second integration by parts yields

$$\ldots = -\frac{1}{y^2}\cos(x)\exp(-yx)\Big|_0^\infty - \frac{1}{y^2}\int \mathrm{d}x \, \sin(x)\exp(-yx) \quad \text{(X.532)}$$

where the first term in this case yields -1 at the lower integration boundary. Collecting the terms gives

$$\left(1 + \frac{1}{y^2}\right)\int_0^\infty \mathrm{d}x \, \sin(x)\exp(-yx) = \frac{1}{y^2} \quad \text{(X.533)}$$

such that

$$\int_0^\infty \mathrm{d}x \, \sin(x)\exp(-yx) = \frac{\frac{1}{y^2}}{1 + \frac{1}{y^2}} = \frac{1}{1 + y^2} \quad \text{(X.534)}$$

and finally

$$\int_0^\infty \mathrm{d}x\,\frac{\sin(x)}{x} = \int_0^\infty \mathrm{d}y\,\frac{1}{1+y^2} = \arctan(x)\Big|_0^\infty = \frac{\pi}{2} \tag{X.535}$$

The inverse problem and slight generalisation of the above calculation is the Fourier-transform of $1/r$,

$$\int_0^\infty r^2\mathrm{d}r \int_{-1}^{+1}\mathrm{d}\mu \int_0^{2\pi}\mathrm{d}\varphi\,\frac{1}{r}\exp(-\mathrm{i}kr\mu) \quad\rightarrow\quad \int_0^\infty r^2\mathrm{d}r \int_{-1}^{+1}\mathrm{d}\mu \int_0^{2\pi}\mathrm{d}\varphi\,\frac{\exp(-\lambda r)}{r}\exp(-\mathrm{i}kr\mu) \tag{X.536}$$

where the issue about convergence of the integral can be alleviated by introducing a factor $\exp(-\lambda r)$ to the integrand, and by considering the limit $\lambda \to 0$ after the integration: This method is known as regularisation of an integral. Physically, we compute the Fourier-transform of a Yukawa-potential instead of a Coulomb-potential. Continuing as before gives

$$\ldots = 4\pi\int_0^\infty r^2\mathrm{d}r\,\frac{\exp(-\lambda r)}{r}\frac{\sin(kr)}{kr} = \frac{4\pi}{k}\int_0^\infty \mathrm{d}r\,\exp(-\lambda r)\sin(kr) \tag{X.537}$$

The remaining integral can be solved again by double integration by parts: Firstly,

$$\int_0^\infty \mathrm{d}r\,\exp(-\lambda r)\sin(kr) = -\frac{1}{\lambda}\exp(-\lambda r)\sin(kr)\Big|_0^\infty + \frac{k}{\lambda}\int_0^\infty \mathrm{d}r\,\exp(-\lambda r)\cos(kr) \tag{X.538}$$

where the first term vanishes at both boundaries. Applying the second integration by parts on the remaining term yields

$$\frac{k}{\lambda}\int_0^\infty \mathrm{d}r\,\exp(-\lambda r)\cos(kr) = -\frac{k}{\lambda^2}\exp(-\lambda r)\cos(kr)\Big|_0^\infty - \frac{k^2}{\lambda^2}\int_0^\infty \mathrm{d}r\,\exp(-\lambda r)\sin(kr) \tag{X.539}$$

where at this step the first term vanishes at the upper, but not at the lower boundary. Consequently,

$$\left(1+\frac{k^2}{\lambda^2}\right)\int_0^\infty \mathrm{d}r\,\exp(-\lambda r)\sin(kr) = \frac{k}{\lambda^2} \tag{X.540}$$

suggesting for the final result:

$$\frac{4\pi}{k}\int_0^\infty \mathrm{d}r\,\exp(-\lambda r)\sin(kr) = 4\pi\frac{\frac{1}{\lambda^2}}{1+\frac{k^2}{\lambda^2}} = \frac{4\pi}{k^2+\lambda^2} \rightarrow \frac{4\pi}{k^2} \quad \text{for} \quad \lambda\to 0 \tag{X.541}$$

Clearly, the inverse Fourier-transform of $4\pi/k^2$ should be $1/r$ (in three dimensions); as well in agreement with our experience with electrostatic potentials $\Phi \propto 1/r$ around point charges. The regularisation

$$\frac{1}{r} \to \frac{\exp(-\lambda r)}{r} \quad \text{corresponds to} \quad \frac{4\pi}{k^2} \to \frac{4\pi}{k^2+\lambda^2}, \tag{X.542}$$

and would work for inverse Fourier-transforms just as well.

A more professional method, which generalises to other types of Green-functions more easily, is to use the residue theorem from complex analysis. Restarting at

$$G(r) = \int_0^\infty \frac{k^2 \mathrm{d}k}{(2\pi)^3} \int_{-1}^{+1} \mathrm{d}\mu \int_0^{2\pi} \mathrm{d}\varphi\, \frac{4\pi}{k^2} \exp(+\mathrm{i}kr\mu) = \frac{2}{\pi} \int_0^\infty \mathrm{d}k\, \frac{\sin(kr)}{kr} \tag{X.543}$$

led us to the dk-integration over the spherical Bessel function. We can extend the integration domain from $-\infty$ to $+\infty$ as the integrand is a symmetric function, and write $\sin(x)$ out in terms of complex exponentials:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \frac{\sin x}{x} = \frac{1}{2\mathrm{i}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x} \left(\exp(\mathrm{i}x) - \exp(-\mathrm{i}x)\right) \to \frac{1}{2\mathrm{i}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{z} \left(\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)\right) \tag{X.544}$$

by continuation to the complex plane. The two terms need to be treated differently when closing the integration to a loop: The first term $\exp(\mathrm{i}z)$ will decrease exponentially towards the positive imaginary axis, so one should close the integration contour there, while the second term $\exp(-\mathrm{i}z)$ decreases exponentially towards the negative imaginary axis, so this is where the loop should be closed. Keep in mind that the first loop is traversed in the mathematically positive sense, while the second one in the negative sense, leading in principle to negative results. Now, the integrand needs to get shifted by $\pm\mathrm{i}\epsilon$ with a small $\epsilon > 0$, such that the pole is contained in one of the integration contours and does not lie on the real axis. Let's chose to move the integrand towards the positive imaginary axis by changing z to $z - \mathrm{i}\epsilon$. In this case, only the first term contributes to the integral (with the integration contour $\frown$) as the second integration contour ($\smile$) does not contain the pole and is therefore zero:

$$\frac{1}{2\mathrm{i}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}z}{z} \left(\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)\right) = \frac{1}{2\mathrm{i}} \oint_{\frown} \frac{\mathrm{d}z}{z} \exp(\mathrm{i}z) + \frac{1}{2\mathrm{i}} \oint_{\smile} \frac{\mathrm{d}z}{z} \exp(-\mathrm{i}z) \tag{X.545}$$

Simplifying the relation further, the loop-integral can be solved with Cauchy's integral formula:

$$\oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta)}{\zeta - z} = 2\pi\mathrm{i}\, g(z) \tag{X.546}$$

with ζ set to zero. As $\exp(\mathrm{i}\zeta) = 1$ at this location, the sought integral bocomes

$$2\int_0^\infty \mathrm{d}x\, \frac{\sin x}{x} = \int_{-\infty}^{+\infty} \mathrm{d}x\, \frac{\sin x}{x} = \frac{1}{2\mathrm{i}} \oint_{\frown} \frac{\mathrm{d}z}{z} \exp(\mathrm{i}z) = \pi. \tag{X.547}$$

Fig. 34 illustrates the integrand of the Green-function for Δ in Fourier-space, with the singularity at the origin.

While these methods generalise straightforwardly to $n \geq 4$, the case of $n = 2$ is downright weird. The corresponding Poisson-equation reads

$$\Delta\Phi = -2\pi\rho \quad \text{in } \textit{two} \text{ dimensions,} \tag{X.548}$$

because the solid angle element in 2d is 2π, as the circumference of a circle with unit radius. But the Fourier-transform of Δ is still $\propto 1/k^2$ as shown before, only that $k^2 = k_x^2 + k_y^2$ in 2 dimensions. Writing formally

$$G(\boldsymbol{r}) = 2\pi \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2} \exp(i\boldsymbol{k} \cdot \boldsymbol{r}) = \int_0^\infty k dk \int_0^{2\pi} d\varphi \, \frac{1}{k^2} \exp(ikr\cos\varphi) \tag{X.549}$$

after introducing polar coordinates that imply $d^2k = kdkd\varphi$, and writing the scalar product as $\boldsymbol{k} \cdot \boldsymbol{r} = kr\cos\varphi$, with φ being the angle between $\boldsymbol{k}$ and $\boldsymbol{r}$. Carrying out the $d\varphi$-integration first leads to the cylindrical Bessel-function J_0, because

$$J_0(kr) = \int_0^{2\pi} d\varphi \, \exp(ikr\cos\varphi) \tag{X.550}$$

such that

$$G(\boldsymbol{r}) = \int_0^\infty \frac{dk}{k} J_0(kr) \rightarrow \int_0^\infty dk \, \frac{k}{k^2 + \lambda^2} J_0(kr) \tag{X.551}$$

by introducing a regularisation in the denominator, which avoids the divergence at $k = 0$. Integrations of this type have the general solution

$$\int_0^\infty dk \, \frac{k^{\nu+1}}{(k^2+\lambda^2)^{\mu+1}} J_\nu(kr) = \frac{r^\mu \lambda^{\nu-\mu}}{2^\mu \Gamma(\mu+1)} K_{\nu-\mu}(\lambda r) = K_0(r\lambda) \quad \text{with } \nu = \mu = 0 \tag{X.552}$$

in our particular case, with $K_0(r\lambda)$ being the modified Bessel-function of the second kind,

$$K_0(r\lambda) = \int_0^\infty dt \frac{\cos(r\lambda t)}{\sqrt{1+t^2}}. \tag{X.553}$$

This particular Bessel-function can be written in terms of a power series in its argument $r\lambda$. In the limit of vanishing regularisation, the value of the power series is dominated by its first term:

$$K_0(r\lambda) = -\left(\ln(r\lambda) + \gamma\right) I_0(r\lambda) \tag{X.554}$$

with $I_0(r\lambda)$ as the modified Bessel function of the first kind approaching unity in the limit $\lambda \rightarrow 0$, leaving $G(r) \propto \ln(r)$. γ is Euler's constant.

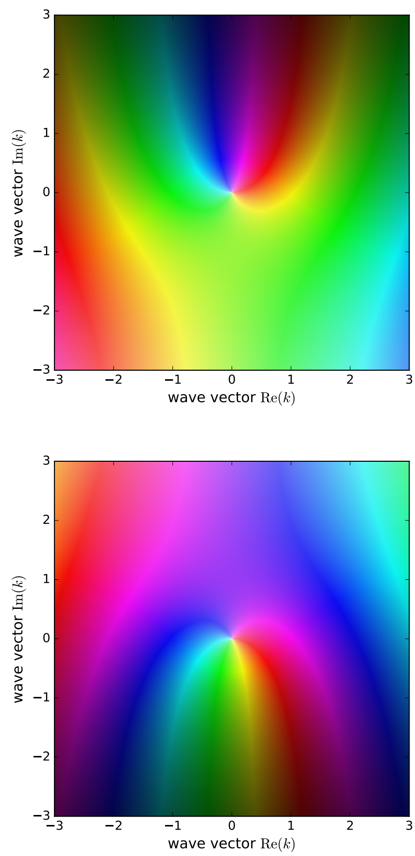


Figure 34: Function $\exp(\pm \mathrm{i}k)/k$ *over the complex plane* $k = \mathrm{Re}(k) + \mathrm{i}\,\mathrm{Im}(k)$*, with color indicating phase and hue indicating the absolute value, for the positive sign the exponent (decreasing towards the positive imaginary axis) on the top and the negative sign (decreasing towards the negative imaginary axis) on the bottom. The singularity at the origin is clearly visible.*

X.5 *Spectra of musical instruments*

An externally driven oscillator illustrates nicely the purpose of a Green-function to cope with inhomogeneities: Let's work with a harmonic oscillator with proper frequency ω_0, a damping γ driven by an external acceleration $a(t)$. Its defining differential equation is

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x(t) = a(t) \tag{X.555}$$

Finding a solution for the homogeneous equation is straightforward: The ansatz $x(t) \propto \exp(\mathrm{i}\omega t)$ yields the characteristic equation $\omega^2 - \mathrm{i}\omega\gamma - \omega_0^2 = 0$, with two solutions, $\omega_\pm = \left(\mathrm{i}\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}\right)/2$. Effectively, this corresponds to taking the Fourier-transform of the differential equation, which then becomes algebraic:

$$\int \frac{\mathrm{d}\omega}{2\pi} \left[-\omega^2 + \mathrm{i}\gamma\omega + \omega_0^2\right] \exp(\mathrm{i}\omega t) x(\omega) = 0 \tag{X.556}$$

as the differentiation $\mathrm{d}/\mathrm{d}t$ replaces the prefactor $\mathrm{i}\omega$, such that we recover the quadratic characteristic equation again. The incorporation of the inhomogeneity can easily be achieved in Fourier-space:

$$\int \frac{\mathrm{d}\omega}{2\pi} \left[-\omega^2 + \mathrm{i}\gamma\omega + \omega_0^2\right] \exp(\mathrm{i}\omega t) x(\omega) = \int \frac{\mathrm{d}\omega}{2\pi} a(\omega) \exp(\mathrm{i}\omega t). \tag{X.557}$$

Because the differential equation has become algebraic, solving for $x(\omega)$ is easy:

$$x(\omega) = \frac{1}{-\omega^2 + \mathrm{i}\gamma\omega + \omega_0^2} a(\omega) = \mathrm{G}(\omega) a(\omega) \tag{X.558}$$

such that the inverse Fourier-transform yields $x(t)$ for a given driving term $a(t)$. The product relation in Fourier-space must be a convolution in real space,

$$x(t) = \int \frac{\mathrm{d}\omega}{2\pi} \frac{1}{-\omega^2 + \mathrm{i}\gamma\omega + \omega_0^2} a(\omega) \exp(\mathrm{i}\omega t) = \int \mathrm{d}t'\, \mathrm{G}(t - t') a(t') \tag{X.559}$$

where the inverse differential operator is just the Green-function for this problem:

$$\mathrm{G}(t - t') = \int \frac{\mathrm{d}\omega}{2\pi} \frac{1}{-\omega^2 + \mathrm{i}\gamma\omega + \omega_0^2} \exp(\mathrm{i}\omega(t - t')) \tag{X.560}$$

$\mathrm{G}(\omega)$ or equivalently, $\mathrm{G}(t - t')$ determines the response of the system, i.e. the damped harmonic oscillator, to an external driving. Most obviously, this is understood in Fourier-space, where $\mathrm{G}(\omega)$ translates the driving $a(\omega)$ to the resulting amplitude $x(\omega)$, frequency by frequency. In configuration space, $\mathrm{G}(t - t')$ is likewise the response of the system, and it is defined formally as the solution to the differential equation to a δ_D-like inhomogeneity,

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right) \mathrm{G}(t - t') = \delta_D(t - t') \tag{X.561}$$

because any inhomogeneity can be constructed from this by linear superposition: Multiplying both sides with $a(t')$ and integrating over $\mathrm{d}t'$ gives

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right)\underbrace{\int \mathrm{d}t'\, G(t-t')a(t')}_{=x(t)} = \int \mathrm{d}t'\delta_D(t-t')a(t') = a(t) \tag{X.562}$$

such that the solution for the amplitude as a function of time has to be given by

$$x(t) = \int \mathrm{d}t'\, G(t-t')a(t') \tag{X.563}$$

i.e. as a convolution relation over the excitation $a(t)$. The interpretation of the response $G(t-t')$ as defined by eqn. (X.561) would now be the solution to the dynamical system to an infinitely sharp excitation. Actually, this is sensible, as it would in fact contain all possible Fourier-modes, even at equal amplitude. But is it possible to construct the Green-function explicitly from the differential operator? After all, the inhomogeneity $a(t)$ is taken care of by the integration eqn. (X.563) and the Green-function itself is defined formally by eqn. (X.561): In fact, in Fourier-space this relation reads:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right)G(t-t') = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}}{\mathrm{d}t} + \omega_0^2\right)\int\frac{\mathrm{d}\omega}{2\pi}\,G(\omega)\exp(\mathrm{i}\omega t) =$$
$$\int\frac{\mathrm{d}\omega}{2\pi}\left(-\omega^2 + \mathrm{i}\gamma\omega + \omega_0^2\right)G(\omega)\exp(\mathrm{i}\omega t) = \int\frac{\mathrm{d}\omega}{2\pi}\exp(\mathrm{i}\omega t) = \delta_D(t-t') \tag{X.564}$$

such that

$$G(\omega) = \frac{1}{-\omega^2 + \mathrm{i}\gamma\omega + \omega_0^2} \tag{X.565}$$

with the inverse transform

$$G(t-t') = \int\frac{\mathrm{d}\omega}{2\pi}\,G(\omega)\exp(\mathrm{i}\omega(t-t')) = \int\frac{\mathrm{d}\omega}{2\pi}\,\frac{1}{-\omega^2 + \mathrm{i}\gamma\omega + \omega_0^2}\exp(\mathrm{i}\omega(t-t')) \tag{X.566}$$

which can be shown to be a Lorentzian spectral line profile.

To complete the analogy to electrodynamics it's instructive to think of the inhomogeneity ρ in electrostatic Poisson-equation $\Delta\Phi = -4\pi\rho$ as the external driving that perturbs the solution to the Laplace equation $\Delta\Phi = 0$. The resulting Green-function $G(\omega)$ is complex-valued; its real and imaginary parts are depicted in Fig. 35, along with its modulus and phase.

A more complete view is presented in Fig. 36, where the Green-function is shown with the phase in color and the modulus in hue.

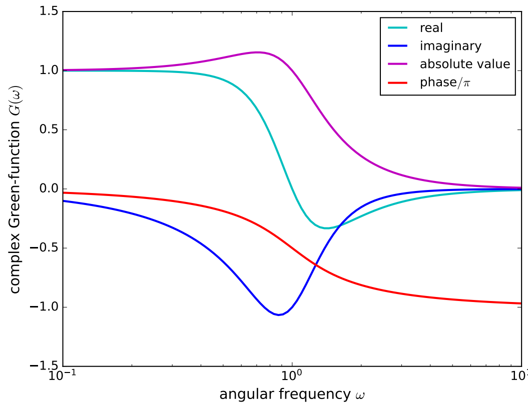


Figure 35: Complex-valued Green-function $\mathrm{G}(\omega)$ *for the damped harmonic oscillator, for* $\omega_0 = \gamma = 1$*, specifically the real and imaginary parts as well as the modulus and the phase angle.*

An external, sinusoidal driving would correspond to a choice of a value for ω on the real axis, and a value close to the two singularities would result in resonant driving. The singularities are situated at

$$\omega^2 - \mathrm{i}\gamma\omega - \omega_0^2 = 0 \quad \rightarrow \quad \omega_\pm = \frac{\mathrm{i}\gamma \pm \sqrt{4\omega_0^2 - \gamma^2}}{2}, \tag{X.567}$$

i.e. at $\sqrt{3}/2 + \mathrm{i}/2$ for the numerical example with $\omega_0 = \gamma = 1$.

Fig. 37 shows spectra for a range of musical instruments. All spectra show the harmonic series of integer multiples of the base note. Their relative amplitudes determine the sound of the respective instruments.

Fig. 38 illustrates, how incredibly well-fitting the Lorentzian line shape for spectra lines actually is. From this observation, one might conclude that a damped harmonic oscillator with an external driving is a good mechanical model for the sound generation in a musical instrument, and motivates sound engineering in a synthesiser.

X.6 *Spherical harmonics*

It is well possible to construct complete orthonormal systems of functions on other manifolds, for instance on the surface of a sphere. As in the case of plane waves for Euclidean space with Cartesian coordinate, which solve the Helmholtz differential equation, one can look for the set of solutions to the wave equation

$$\Delta Y_{\ell m}(\theta,\varphi) = -\ell(\ell+1) Y_{\ell m}(\theta,\varphi) \quad \rightarrow \quad \left[\Delta + \ell(\ell+1)\right] Y_{\ell m}(\theta,\varphi) = 0 \tag{X.568}$$

where the Laplace-operator is a differentiation with respect to the angular coordinate θ and φ. Comparing to the Cartesian Helmholtz-PDE $\left[\Delta + k^2\right]\exp(\pm k_i r^i) = 0$ one identifies the term $\ell(\ell+1)$ with k^2, implying that π/ℓ should be a wave length (in terms of radians) just like $2\pi/k$ would be a physical wave length λ.

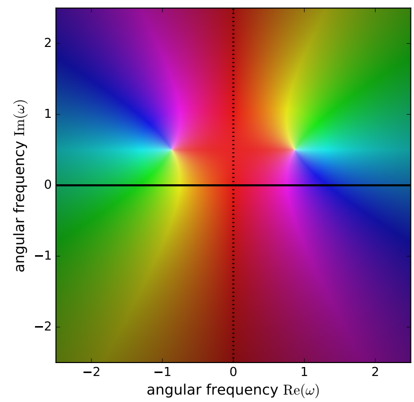


Figure 36: Complex-valued Green-function $G(\omega)$ *over the complex plane* $\omega = \mathrm{Re}(\omega) + \mathrm{i}\,\mathrm{Im}(\omega)$*, with phase indicated by colour and absolute value by hue, again for* $\omega_0 = \gamma = 1$.

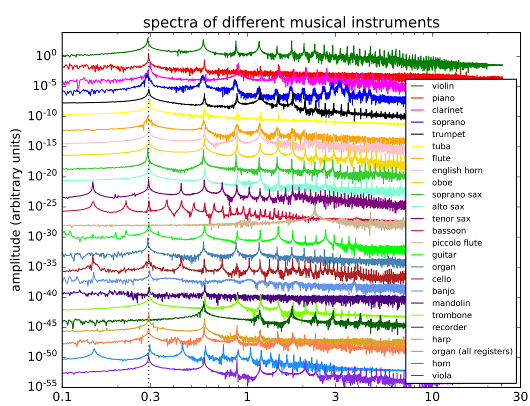


Figure 37: Spectra of different musical instruments, showing higher-order harmonics.

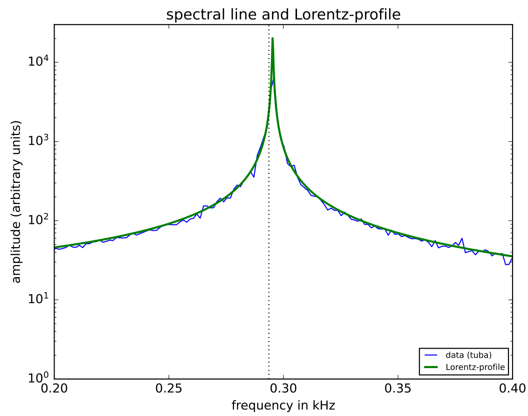


Figure 38: Spectral line of a tone with a best-fitting Lorentz-profile.

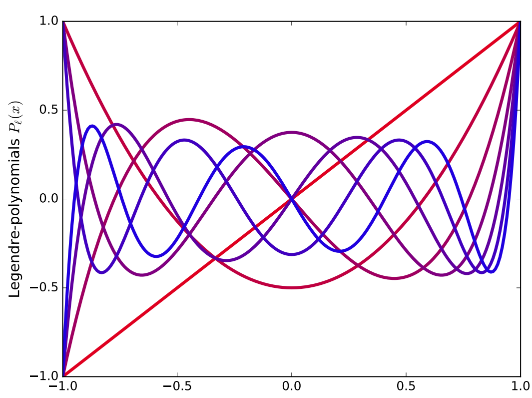


Figure 39: Legendre polynomials $P_\ell(x)$ for $\ell = 1 \ldots 8$, with even parity for even ℓ, and odd parity for odd ℓ.

The Laplace-operator Δ in angular coordinates applied onto a scalar function $\psi(\theta, \varphi)$ reads explicitly

$$\Delta\psi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} \tag{X.569}$$

As there are no mixed derivatives one should try a separation ansatz

$$\psi(\theta, \varphi) = T(\theta)P(\varphi) \tag{X.570}$$

so that the Helmholtz-PDE becomes

$$\Delta\psi = \frac{P(\varphi)}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T(\theta)}{\partial\theta}\right) + \frac{T(\theta)}{\sin^2\theta}\frac{\partial^2 P(\varphi)}{\partial\varphi^2} = -\ell(\ell+1)T(\theta)P(\varphi) \tag{X.571}$$

such that division by $T(\theta)P(\varphi)$ separates the terms as dependent on θ or φ

$$\frac{\sin\theta}{T}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \ell(\ell+1)\sin^2\theta = -\frac{1}{P}\frac{\partial^2 P}{\partial\varphi^2} \tag{X.572}$$

to the left and right side of the equation: They must therefore both be equal to a separation constant m^2. Then, the right side gives

$$\frac{1}{P}\frac{\partial^2 P}{\partial\varphi^2} = -m^2 \quad \rightarrow \quad \left(\frac{\partial^2}{\partial\varphi^2} + m^2\right)P(\varphi) = 0 \tag{X.573}$$

which is again a Helmholtz-differential equation, this time in φ only. It has wave-type solutions

$$P(\varphi) \propto \exp(\pm \mathrm{i}m\varphi) \tag{X.574}$$

with m playing the role of a wave number, but it has to be integer because otherwise the continuity of the solution could not be ensured when rotating by 2π:

$$P(\varphi+2\pi) = P(\varphi) \quad \text{implies} \quad \exp(\pm \mathrm{i}m(\varphi+2\pi)) = \underbrace{\exp(\pm 2\pi \mathrm{i}m)}_{=1}\exp(\pm \mathrm{i}m\varphi) = \exp(\pm \mathrm{i}m\varphi) \tag{X.575}$$

if m is integer. With this knowledge we return to the θ-equation, which becomes the associated Legendre-differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta} + \ell(\ell+1)\right]T(\theta) = 0 \tag{X.576}$$

after resorting the terms, where the particular case $m = 0$ leads to the actual Legendre-differential equation,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \ell(\ell+1)T(\theta) = 0 \tag{X.577}$$

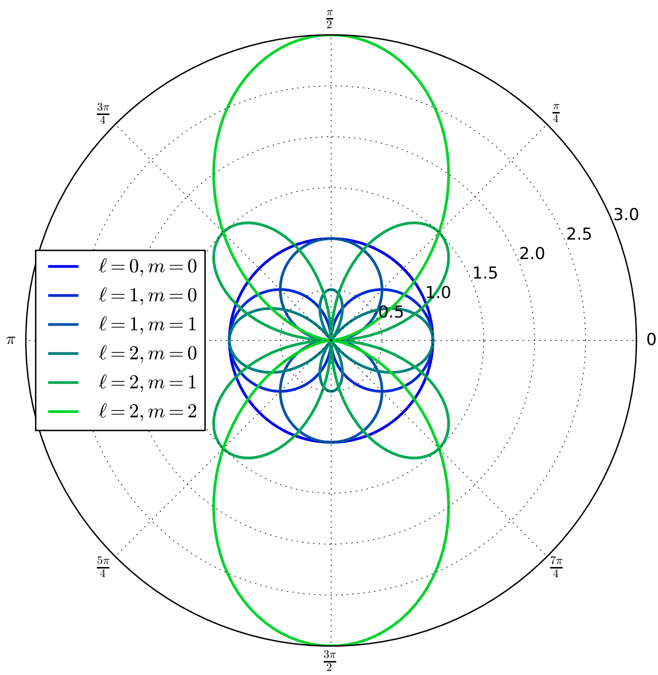


Figure 40: Associated Legendre polynomials $P_{\ell m}(\cos\theta)$ *in a polar representation.*

Transitioning to the new variable $x = \cos\theta$ with $\sin\theta = \sqrt{1-x^2}$ then yields

$$(1-x^2)\frac{d^2T}{dx^2} - 2x\frac{dT}{dx} + \ell(\ell+1)T(x) = 0 \tag{X.578}$$

whose solution are the Legendre-polynomials $P_\ell(x)$. They can be shown to obey an orthogonality relation

$$\int_{-x}^{+1} dx\, P_\ell(x)P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'} \tag{X.579}$$

in the same way as the plane waves $\exp(\pm i m\varphi)$ for the azimuthal coordinate, confirming that the Helmholtz differential equation in fact defines a system of orthonormal waves on the surface of the sphere.

In the same way there is an orthogonality relation for the solutions to the associated Legendre differential equation

$$\int_{-1}^{+1} \mathrm{d}x\; P_{\ell m}(x) P_{\ell' m'}(x) = \frac{2}{2\ell+1}\frac{(\ell+|m|)!}{(\ell-|m|)!}\delta_{\ell\ell'}\delta_{mm'} \tag{X.580}$$

such that the definition of the spherical harmonics including the prefactors

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{4\pi}{2\ell+1}}\sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}P_{\ell m}(\cos\theta)\exp(+\mathrm{i}m\varphi) \tag{X.581}$$

gives the fundamental orthogonality

$$\int_{4\pi} \mathrm{d}\Omega\; Y_{\ell m}(\theta,\varphi) Y^*_{\ell' m'}(\theta,\varphi) = \delta_{\ell,\ell'}\delta_{mm'} \tag{X.582}$$

and completeness relations

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell} Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}(\theta',\varphi') = \delta_D(\theta-\theta')\delta_D(\varphi-\varphi') \tag{X.583}$$

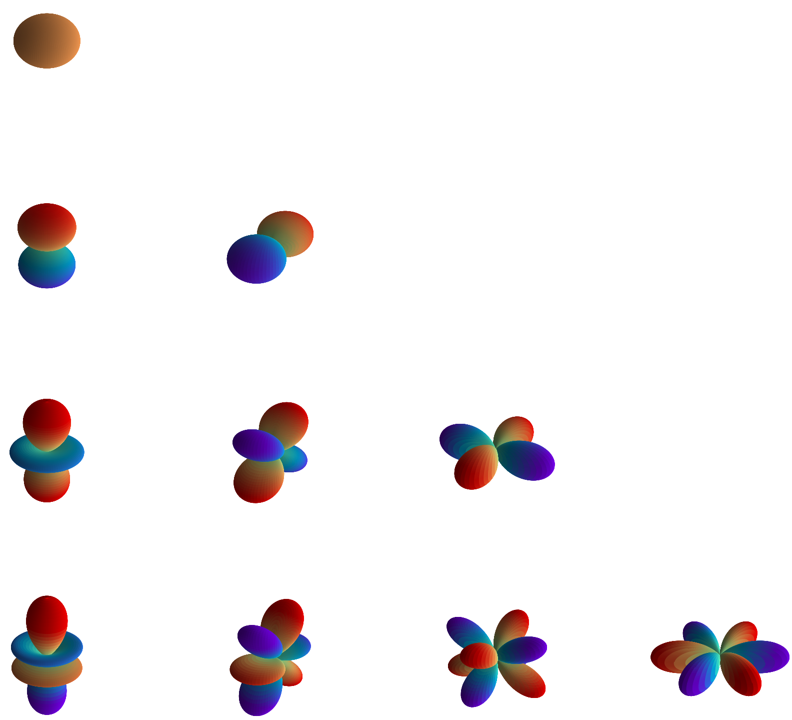


Figure 41: Spherical harmonics $Y_{\ell m}(\theta, \varphi)$ *for* $\ell = 0, 1, 2, 3$ *(top to bottom) and* $0 \leq m \leq \ell$ *(corresponding rows).*

Y COMPLEX CALCULUS

Y.1 *Aspects of complex differentiability*

Many of the integrals needed for the construction of a Green-function with the Fourier method are not solvable with elementary methods, i.e. integration by substitution, by parts or using partial fractions, for instance

$$\int_{-\infty}^{+\infty} \mathrm{d}\omega \, \frac{1}{(ck)^2 - \omega^2} \exp\left(-\mathrm{i}\omega(t - t')\right) \tag{Y.584}$$

which shows two singularities at $\omega = \pm ck$. Methods from complex analysis, though, provide a pathway of doing that.

A function $g(z) = u(x, y) + \mathrm{i}v(x, y)$ maps a complex argument $z = x + \mathrm{i}y$ onto a complex value $g = u + \mathrm{i}y$. It is continuous in ζ if there is an $\epsilon > 0$ for every $\delta > 0$ such that $|g(z) - g(\zeta)| < \epsilon$ follows form $|z - \zeta| < \delta$. In other words, the limit

$$\lim_{\zeta \to z} |g(z) - g(\zeta)| = 0 \tag{Y.585}$$

does not depend on the way how ζ approaches z. The function $g(z)$ is complex differentiable in z, if the limit

$$\lim_{\zeta \to z} \frac{g(z) - g(\zeta)}{z - \zeta} = \frac{\mathrm{d}g}{\mathrm{d}z}(z) \tag{Y.586}$$

exists and is unique, or in other words: if the differential quotient is continuous.

Complex differentiability is a weird and very powerful concept. Historically, four different aspects have been discovered which turn out to be identical and merely different sides of the same idea: (i) complex differentiable, (ii) analytical, meaning that the Cauchy-Riemann differential equations hold, (iii) regular, defined as a vanishing loop integral over closed curves, and (iv) holomorphic, meaning that the function fulfils the residue theorem. An weidly enough, it blurs the boundaries between integration and differentiation, as exemplified by the Cauchy-theorem. Fundamentally, it is yet another example of the powerful concept of exact differentials.

Y.2 *Cauchy-Riemann differential equations*

In a complex differentiable function, the derivative does not depend on the direction how Δz, itself a complex number, approaches zero,

$$\frac{\mathrm{d}g}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z}. \tag{Y.587}$$

Therefore, the derivative in x-direction parallel to the real axis,

$$\lim_{\Delta x \to 0} \frac{g(z + \Delta x) - g(z)}{\Delta x} = \frac{\partial g}{\partial x} \tag{Y.588}$$

and the derivative in the y-direction parallel to the imaginary axis,

$$\lim_{\Delta y\to 0}\frac{g(z+\mathrm{i}\Delta y)-g(z)}{\mathrm{i}\Delta y}=\frac{1}{\mathrm{i}}\frac{\partial g}{\partial y} \tag{Y.589}$$

must be equal. Writing this relation in terms of the components of g yields

$$\frac{\partial g}{\partial x}=\frac{\partial u}{\partial x}+\mathrm{i}\frac{\partial v}{\partial x}=\frac{1}{\mathrm{i}}\frac{\partial g}{\partial y}=\frac{1}{\mathrm{i}}\frac{\partial u}{\partial y}+\frac{\partial g}{\partial y} \tag{Y.590}$$

and with a subsequent separation of the real and imaginary parts on arrives at the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial x}=+\frac{\partial v}{\partial y}\quad\text{and}\quad\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}. \tag{Y.591}$$

The notions of complex differentiability and the fulfilment of the Cauchy-Riemann differential equations is equivalent.

Y.3 *Complex line and loop integrals*

Given a curve Γ parameterised with λ running from point A with coordinates $z(a)$ to point B at $z(b)$, one can define a complex line integral by reducing it to an integral over the parameter by substitution,

$$\int_{\Gamma_{\mathrm{AB}}}\mathrm{d}z\,g(z)=\int_a^b\mathrm{d}\lambda\,\frac{\mathrm{d}z}{\mathrm{d}\lambda}g(z(\lambda)). \tag{Y.592}$$

Covering the same path in opposite direction yields the same numerical result, but with a negative sign

$$\int_{\Gamma_{\mathrm{BA}}}\mathrm{d}z\,g(z)=\int_b^a\mathrm{d}\lambda\,\frac{\mathrm{d}z}{\mathrm{d}\lambda}g(z(\lambda))=-\int_a^b\mathrm{d}\lambda\,\frac{\mathrm{d}z}{\mathrm{d}\lambda}g(z(\lambda)) \tag{Y.593}$$

If an integral does not depend on the particular path from A to B, one can assemble a trip from A to B on one path followed by a return trip from B to A on another path, with the two contributions cancelling each other, with the overall result being

$$\int_{\Gamma_{\mathrm{AB}}}\mathrm{d}z\,g(z)+\int_{\Gamma_{\mathrm{BA}}}\mathrm{d}z\,g(z)=\oint_{\Gamma}\mathrm{d}z\,g(z)=0 \tag{Y.594}$$

Just as before, traversing a closed loop in the opposite sense of rotation would yield an overall minus sign. Writing this relation component-wise

$$\oint_{\Gamma}\mathrm{d}z\,g(z)=\oint_{\Gamma}(\mathrm{d}x+\mathrm{i}\mathrm{d}y)\,(u+\mathrm{i}v)=\oint_{\Gamma}(u\mathrm{d}x-v\mathrm{d}y)+\mathrm{i}\oint_{\Gamma}(v\mathrm{d}x+u\mathrm{d}y)=0 \tag{Y.595}$$

Both terms can be reformulated as area integrals by virtue of Green's theorem, $\partial\mathcal{C} = \Gamma$,

$$\oint_\Gamma (u\mathrm{d}x - v\mathrm{d}y) = -\int_\mathcal{C} \mathrm{d}x\mathrm{d}y \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \quad \text{and} \quad \oint_\Gamma (v\mathrm{d}x + u\mathrm{d}y) = \int_\mathcal{C} \mathrm{d}x\mathrm{d}y \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{Y.596}$$

where one immediately recognises the Cauchy-Riemann equations in the integrands, making both results vanish. In summary,

$$\oint_\Gamma \mathrm{d}z\, g(z) = 0 \tag{Y.597}$$

for any complex differentiable function. ✍ Green's theorem, which allows the conversion of a loop integral to an area integral works for simply connected regions.

Y.4 *Residue theorem and holomorphic functions*

The Cauchy-theorem states that every value of a complex differentiable function inside a closed curve Γ is fixed by the values on that curve,

$$g(z) = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta)}{\zeta - z} \tag{Y.598}$$

Functions with that property are called holomorphic, which is synonymous to complex differentiable. In fact,

$$\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta)}{\zeta - z} = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta) \overbrace{-g(z) + g(z)}^{=0}}{\zeta - z} =$$
$$\frac{g(z)}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{1}{\zeta - z} + \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta) - g(z)}{\zeta - z} = g(z), \tag{Y.599}$$

after reordering the terms and using that $\oint \mathrm{d}\zeta\, g(z) \ldots = g(z) \oint \mathrm{d}\zeta \ldots$. The first term can be shown to be

$$\oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta} = \oint_\Gamma \mathrm{d}\ln\zeta = \int_0^{2\pi} \mathrm{d}\lambda\, \frac{\mathrm{d}\zeta}{\mathrm{d}\lambda}\frac{1}{\zeta} = \mathrm{i}\int_0^{2\pi} \mathrm{d}\lambda\, \exp(\mathrm{i}\lambda)\exp(-\mathrm{i}\lambda) = \mathrm{i}\int_0^{2\pi} \mathrm{d}\lambda = 2\pi\mathrm{i} \tag{Y.600}$$

after substitution $\zeta - z \to \zeta$, which can then be solved by choosing the unit circle $\zeta = \exp(\mathrm{i}\lambda)$ with $\mathrm{d}\zeta = \mathrm{i}\exp(\mathrm{i}\lambda)\mathrm{d}\lambda = \mathrm{i}\zeta\mathrm{d}\lambda$ as the integration contour. The second integral can be treated like this:

$$\left|\frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta) - g(z)}{\zeta - z}\right| \leq \frac{1}{|2\pi\mathrm{i}|} \left|\oint_\Gamma \mathrm{d}\zeta\, \frac{g(\zeta) - g(z)}{\zeta - z}\right| \leq \frac{\epsilon}{|2\pi\mathrm{i}|} \left|\oint_\Gamma \mathrm{d}\zeta\, \frac{1}{\zeta - z}\right| = \epsilon \tag{Y.601}$$

if the function g is continuous, which is quite obvious as it is already assumed to be complex differentiable: Then, the integration contour can be chosen to be small enough such that $|g(\zeta) - g(z)| < \epsilon$. In addition, the integral was already shown to be $2\pi\mathrm{i}$. Overall, the second integral is bounded by ϵ, and does effectively does not contribute, as ϵ can be chosen to be arbitrarily small.

It is worth memorising the iconic result

$$\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{\mathrm{d}\zeta}{\zeta} = 1, \tag{Y.602}$$

but what about other powers in ζ? Clearly, for both positive and negative n, as long as $n \neq -1$,

$$\oint \mathrm{d}\zeta\,\zeta^n = \int_0^{2\pi}\mathrm{d}\lambda\,\frac{\mathrm{d}\zeta}{\mathrm{d}\lambda}\zeta^n = \mathrm{i}\int_0^{2\pi}\mathrm{d}\lambda\,\exp(\mathrm{i}\lambda)\exp(\mathrm{i}n\lambda) =$$
$$\mathrm{i}\int_0^{2\pi}\mathrm{d}\lambda\,\exp(\mathrm{i}(n+1)\lambda) = \frac{\exp(\mathrm{i}(n+1)\lambda)}{n+1}\Big|_0^{2\pi} = 0 \tag{Y.603}$$

from elementary integration, again with the parameterised unit circle $\exp(\mathrm{i}\lambda)$ as the integration contour. But alternatively, one could argue that the plane waves form an orthonormal system. Therefore, only for $n = -1$ one gets a nonzero result.

The Cauchy-theorem can be generalised to higher-order derivatives: Starting with a Taylor-expansion of $g(\zeta)$ around z,

$$g(\zeta) = g(z) + \frac{\mathrm{d}g}{\mathrm{d}\zeta}\Big|_z(\zeta - z) + \frac{\mathrm{d}^2 g}{\mathrm{d}z^2}\Big|_z\frac{(\zeta - z)^2}{2} + \cdots \tag{Y.604}$$

Using the results from above, one can isolate $g(z)$ from the series by multiplying it with $1/(\zeta - z)$, followed by a loop integration comprising z:

$$\oint_\Gamma \mathrm{d}\zeta\,\frac{g(\zeta)}{\zeta - z} = g(z)\underbrace{\oint_\Gamma \mathrm{d}\zeta\,\frac{1}{\zeta - z}}_{=2\pi\mathrm{i}} + \frac{\mathrm{d}g}{\mathrm{d}z}\Big|_z\underbrace{\oint_\Gamma \mathrm{d}\zeta\,\frac{\zeta - z}{\zeta - z}}_{=0} + \frac{1}{2}\frac{\mathrm{d}^2 g}{\mathrm{d}z^2}\Big|_z\underbrace{\oint_\Gamma \mathrm{d}\zeta\,\frac{(\zeta - z)^2}{\zeta - z}}_{=0} + \cdots \tag{Y.605}$$

For accessing a higher order derivative, for instance $\mathrm{d}g/\mathrm{d}z$, one would need to multiply the series by $1/(\zeta - z)^2$ before integrating,

$$\oint_\Gamma \mathrm{d}\zeta\,\frac{g(\zeta)}{(\zeta - z)^2} = g(z)\underbrace{\oint_\Gamma \mathrm{d}\zeta\,\frac{1}{(\zeta - z)^2}}_{=0} + \frac{\mathrm{d}g}{\mathrm{d}z}\Big|_z\underbrace{\oint_\Gamma \mathrm{d}\zeta\,\frac{\zeta - z}{(\zeta - z)^2}}_{=2\pi\mathrm{i}} + \frac{1}{2}\frac{\mathrm{d}^2 g}{\mathrm{d}z^2}\Big|_z\underbrace{\oint_\Gamma \mathrm{d}\zeta\,\frac{(\zeta - z)^2}{(\zeta - z)^2}}_{=0} + \cdots \tag{Y.606}$$

This pattern generalises to the Cauchy-theorem for derivatives of $g(z)$,

$$\left.\frac{\mathrm{d}^n g}{\mathrm{d}z^n}\right|_z = \frac{n!}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\zeta \, \frac{g(\zeta)}{(\zeta - z)^{n+1}}, \tag{Y.607}$$

with the interesting implication that derivatives of a complex differentiable function can be obtained through an integration process. If a function is complex differentiable once, it is complex differentiable arbitrarily often, in stark contrast to real differentiability.

The Cauchy-theorem can be applied in the solution of real-valued integrals that can not be solved (easily) by means of elementary integration. A classic example of this is

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{1}{1+x^2} = \arctan x\Big|_{-\infty}^{+\infty} = \pi \tag{Y.608}$$

where a solution is only possibly by using the rule of the derivative of the inverse function and trigonometric identities. Instead, one can perform a complex continuation,

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{1}{1+x^2} \to \int_{-\infty}^{+\infty} \mathrm{d}z \, \frac{1}{1+z^2} \tag{Y.609}$$

where x is interpreted as a complex-valued variable z. The denominator has two poles at $z = \pm\mathrm{i}$, allowing a decomposition into partial fractions,

$$\frac{1}{1+z^2} = \frac{1}{(1+z)(1-z)} = \frac{1}{2\mathrm{i}}\left(\frac{1}{z-\mathrm{i}} - \frac{1}{z+\mathrm{i}}\right), \tag{Y.610}$$

and the integration along the real axis from $-\infty$ to $+\infty$ can be extended by an semi-circular arc, which does not contribute to the value of the integrand, as its arc length increases with radius, but the value of the integrand decreases proportional to the squared radius. This arc now makes the integration a complex loop integral, so that we can write

$$\oint_\Gamma \mathrm{d}z \, \frac{1}{z^2+1} = \frac{1}{2\mathrm{i}} \underbrace{\oint_\frown \frac{\mathrm{d}z}{z-\mathrm{i}}}_{=2\pi} - \frac{1}{2\mathrm{i}} \underbrace{\oint_\frown \Gamma \frac{\mathrm{d}z}{z+\mathrm{i}}}_{=0} = \pi \tag{Y.611}$$

because only the pole at $z = +\mathrm{i}$ is contained inside the integration contour.

One would have arrived at exactly the same result if the arc had been closed at the bottom instead of the top: Then, the sense in which the curve Γ is traversed, is inverted, yielding a negative sign:

$$\oint_\Gamma \mathrm{d}z \, \frac{1}{z^2+1} = \frac{1}{2\mathrm{i}} \underbrace{\oint_\smile \frac{\mathrm{d}z}{z-\mathrm{i}}}_{=0} - \frac{1}{2\mathrm{i}} \underbrace{\oint_\smile \frac{\mathrm{d}z}{z+\mathrm{i}}}_{=-2\pi} = \pi \tag{Y.612}$$

as now the other pole at $z = -\mathrm{i}$ is caught by the integration contour.

Y.5 *Laurent-series*

In the example above we have already embedded a function of a single, real-valued variable into the complex plane, and consider it to a (differentiable) mapping between complex numbers. This idea can be generalised in analytical continuations of a complex function $g(z)$, in cases where it is known in a region around z_0 to a second region around z bounded by Δ. There, the Cauchy-relation

$$g(z) = \frac{1}{2\pi i}\oint_\Delta d\zeta\,\frac{g(\zeta)}{\zeta - z} \quad\text{and}\quad \frac{d^n g}{dz^n}\Big|_z = \frac{n!}{2\pi i}\oint_\Delta d\zeta\,\frac{g(\zeta)}{(\zeta - z)^{n+1}} \tag{Y.613}$$

for any Γ circling the point z allows to access the values of g and its derivatives. The function and its derivatives at z_0 can be used to construct a power series that extends from a region around z_0 to z and defines the continuation of the function in this *terra incognita* bounded by Δ.

The function's values inside Δ are fixed by the Cauchy-theorem, and one can assemble an integration path consisting of two concentric loops Γ_1 (with radius r_1) and Γ_2 with radius r_2, joined by two bridges A_1 and A_2. This integration path replaces Δ, as it would result from continuous deformation within the holomorphic region. Then, $g(z)$ can be computed as

$$g(z) = \frac{1}{2\pi i}\oint_{\Gamma_2} d\zeta\frac{g(\zeta)}{\zeta - z} - \frac{1}{2\pi i}\oint_{\Gamma_1} d\zeta\frac{g(\zeta)}{\zeta - z}, \tag{Y.614}$$

because the contributions along A_1 and A_2 cancel each other due to the opposite direction in which they are traversed. Please note that the second loop Γ_1 contributes with a minus sign as the integration path is followed in a clockwise direction, i.e. in the mathematically negative sense. From the two integrals, the secoond one vanishes because of the Cauchy-theorem because z is outside Γ_1, but the first integral gives a non-vanishing result, with z being contained in Γ_2.

In our construction, the values of ζ traversed in the integration along the large loop Γ_2 have a modulus of r_2. Then, one can argue that

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0}\sum_n\left(\frac{z - z_0}{\zeta - z_0}\right)^n \tag{Y.615}$$

where in the last step we replaced the $1/(1-q)$-term with its corresponding geometric series. There is no issue of convergence of

$$\sum_n q^n = \frac{1}{1-q} \quad\text{because}\quad q = \left|\frac{z - z_0}{\zeta - z_0}\right| = \frac{r}{r_2} < 1 \tag{Y.616}$$

Conversely, if ζ is situated on the loop Γ_1 with radius r_1, an analogous argument applies, as

$$\frac{1}{\zeta - z} = \frac{1}{z - z_0}\frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = \frac{1}{z - z_0}\sum_n\left(\frac{\zeta - z_0}{z - z_0}\right)^n. \tag{Y.617}$$

In this case, convergence of the geometric series is ensured by

$$\sum_n p^n = \frac{1}{1-p} \quad \text{where} \quad p = \left|\frac{\zeta - z_0}{z - z_0}\right| = \frac{r_1}{r} < 1 \tag{Y.618}$$

Collecting these results leads to

$$g(z) = \frac{1}{2\pi i}\oint_{\Gamma_2} d\zeta\, g(\zeta) \sum_n \frac{(z-z_0)^n}{(\zeta - z_0)^{n+1}} - \frac{1}{2\pi i}\oint_{\Gamma_1} d\zeta\, g(\zeta) \sum_n \frac{(\zeta - z_0)^n}{(z-z_0)^{n+1}}. \tag{Y.619}$$

It is an interesting realisation that the two fractions are inverses of each other, leading to a natural continuation of the series towards negative n. Reordering integration and summation yields:

$$g(z) = \sum_n \left(\frac{1}{2\pi i}\oint_{\Gamma_2} d\zeta\, \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}\right) \times (z-z_0)^n - \sum_n \left(\frac{1}{2\pi i}\oint_{\Gamma_1} d\zeta\, \frac{g(\zeta)}{(\zeta - z_0)^{-n}}\right) \times (z-z_0)^{-(n+1)}. \tag{Y.620}$$

In summary, this result can be rewritten

$$g(z) = \sum_{n=-\infty}^{+\infty} a_n (z-z_0)^n \quad \text{with} \quad a_n = \frac{1}{2\pi i}\oint_{\Gamma} d\zeta\, \frac{g(\zeta)}{(\zeta - z_0)^{n+1}}, \tag{Y.621}$$

for any close curve running between Γ_1 and Γ_2, where the minus-sign is cancelled by choosing a joint sense of rotation for the integration loop. This result is known as the Laurent-series, a power-law expansion of holomorphic functions, with its remarkable negative powers.

Y.6 *Residue theorem*

Looking at the Laurent series for $g(z)$,

$$\sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots, \tag{Y.622}$$

all terms belonging to positive indices $n \geq 0$ remain finite in the limit $z \to z_0$, while the terms for negative $n \neq 0$ are divergent. The function $g(z)$ would possess a pole of order $-n$ at z_0 if the Laurent series terminates at finite $-n$. Please note that the Laurent-series is constructed in a consistent way: Applying

$$\frac{1}{2\pi i}\oint_{\Gamma} dz\, \ldots \tag{Y.623}$$

to both sides yields for the terms with positive exponents $n \geq 0$,

$$\frac{1}{2\pi i}\oint_{\Gamma} dz\, (z-z_0)^n = 0, \tag{Y.624}$$

and similarly for the negative exponents with $n \geq 2$,

$$\frac{1}{2\pi i}\oint_\Gamma dz\, \frac{1}{(z-z_0)^n} = 0, \tag{Y.625}$$

whereas only the term for $n = -1$ yields a non-vanishing result, namely:

$$\frac{1}{2\pi i}\oint_\Gamma dz\, \frac{1}{z-z_0} = 1. \tag{Y.626}$$

The particular coefficient corresponding to $n = -1$ of the Laurent series,

$$a_{-1} = \frac{1}{2\pi i}\oint_\Gamma d\zeta\, g(\zeta), \tag{Y.627}$$

is called the residue of $g(z)$ at z_0, which needs to be located within Γ.

Y.7 *Conformal mappings*

Analytical (or complex differentiable, or regular, or holomorphic) functions automatically fulfil the Laplace-equation $\Delta g = 0$ in two dimensions and, as such, are viable solutions to the field equation ini vacuum. Starting with $g(z) = u(x,y) + iv(x,y)$ we write:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\underbrace{\frac{\partial u}{\partial v}}_{=\partial v/\partial y} = \frac{\partial}{\partial y}\underbrace{\frac{\partial v}{\partial x}}_{=-\partial u/\partial y} = -\frac{\partial^2 y}{\partial x^2} \quad\rightarrow\quad \Delta u = 0 \tag{Y.628}$$

taking advantage of the fact that partial differentiations interchange and substituting the Cauchy-Riemann equations twice. Conversely, one shows for the imaginary part

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\underbrace{\frac{\partial v}{\partial x}}_{=-\partial u/\partial y} = -\frac{\partial}{\partial y}\underbrace{\frac{\partial u}{\partial x}}_{=\partial v/\partial y} = -\frac{\partial^2 v}{\partial y^2} \quad\rightarrow\quad \Delta v = 0 \tag{Y.629}$$

from which we conclude that $\Delta u + i\Delta v = \Delta(u + iv) = \Delta g = 0$. In addition, as complex conjugation is a linear operation, it is valid that $\Delta g^* = 0$.

Clearly, the solution to the field equation $\Delta\Phi = 0$ in electrostatics in vacuum or to the field equations $\Delta A_i = 0$ for all three components A_i of the vector potential in Coulomb-gauge in magnetostatics, again in vacuum, could be represented by a holomorphic function. One needs to keep in mind, though, that g is a complex number with two components, whereas the potentials are real numbers. Hence the question arises, what the other component Ψ of $g = \Phi + i\Psi$ could represent!

If one were to identify Φ with the real value of g, it would need to represent the electric field $E_i = -\partial\Phi/\partial x^i$ as the gradient of Φ. It is possible to re-express the electric field as a complex number $E_x + iE_y = -\partial\Phi$ with the Wirtinger derivative instead of $E_i = -\partial_i\Phi$ in Cartesian coordinates. Additionally, there seems to be an auxiliary field Ψ, called the stream function, to be identified as $\Psi = v$.

The stream function is always perpendicular to lines of constant potential, which can be seen by this argument: The gradients ∇u and ∇v are clearly perpendicular,

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x} \underbrace{\frac{\partial v}{\partial x}}_{=-\partial u/\partial y} + \frac{\partial u}{\partial y} \underbrace{\frac{\partial v}{\partial y}}_{=+\partial u/\partial x} = 0 \tag{Y.630}$$

by substituting the Cauchy-Riemann differential equations, and so would be the functions Φ and Ψ.

There is a neat shortcut to this relation, by using the tools of Wirtinger-calculus: Motivated by the fact that the coordinates x and y are combined into a complex number $z = x + \mathrm{i}y$ (and its conjugate $z^* = x - \mathrm{i}y$), one can define the composite derivatives:

$$\partial \equiv \frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y} \quad \text{as well as} \quad \partial^* \equiv \frac{\partial}{\partial z^*} = \frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}. \tag{Y.631}$$

Combination of the two derivatives leads directly to the Laplace operator, as both $\partial\partial^* g$ as well as $\partial^*\partial g$ are equal to Δg!

There is a neat application of conformal applications to potentials in vacuum in two dimensions. Commonly, potential problems are easy to solve in highly symmetric charge distributions, which makes the convolution with the Green-function relatively simple: In particular, a convolution of spherical symmetric charge distributions with spherically symmetric Green-functions give rise to the a spherically symmetric potential. To make this point more obvious, let's consider a circularly symmetric charge distribution in two dimensions. The potential is necessarily $\Phi \propto \ln r$ with the electric field $\mathrm{E}_r = 1/r$ and $\mathrm{E}_\varphi = 0$. A more complicated charge distribution would generate the potential $\Phi = \int \mathrm{d}^2 r'\ \rho(\boldsymbol{r}) \ln\left(|\boldsymbol{r} - \boldsymbol{r}'|\right)$, with a potentially complicated $\mathrm{d}^2 r'$-integration.

The problem might be alleviated if a mapping of the old coordinates x, y to new coordinates u, v can be found which would not have an influence on the differential structure of the field equation.

This can in fact be achieved in two dimensions, where the coordinates can be combined into a complex number $z = x + \mathrm{i}y$, for vacuum solutions that obey the Laplace equation $\Delta\Phi = 0$. The Laplace-operator Δ transforms under coordinate change in a peculiar way and acquires just an overall strictly positive, position-dependent prefactor, which is called a conformal factor α^2. The vacuum field equation transforms as $\Delta\Phi \rightarrow \alpha\Delta\Phi = 0$ but clearly, the conformal factor α is irrelevant and drops out for vacuum solutions. Therefore, any vacuum solution in one set of coordinates is automatically a valid vacuum solution in the transformed coordinates. The necessary prerequisite is an analytical coordinate change.

To make things specific, let's consider the mapping

$$\mathrm{G}(u, v) \rightarrow g(x, y) = \mathrm{G}(u(x, y), v(x, y)) \tag{Y.632}$$

and derive the Laplace-equation for g in the coordinates x, y in terms of the Laplace-equation for G in terms of u, v. For the first derivatives one obtains:

$$\frac{\partial g}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial \mathrm{G}}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial \mathrm{G}}{\partial v} \quad \text{as well as} \quad \frac{\partial g}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial \mathrm{G}}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial \mathrm{G}}{\partial v}. \tag{Y.633}$$

Continuing with the second derivatives one arrives at

$$\frac{\partial^2 g}{\partial^2 x} = \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 G}{\partial u^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\frac{\partial^2 G}{\partial u \partial v} + \frac{\partial^2 u}{\partial x^2}\frac{\partial G}{\partial u} + \left(\frac{\partial v}{\partial x}\right)^2 \frac{\partial^2 G}{\partial v^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial x}\frac{\partial^2 G}{\partial v \partial u} + \frac{\partial^2 v}{\partial x^2}\frac{\partial G}{\partial u} \tag{Y.634}$$

together with

$$\frac{\partial^2 g}{\partial^2 y} = \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 G}{\partial u^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\frac{\partial^2 G}{\partial u \partial v} + \frac{\partial^2 u}{\partial x^2}\frac{\partial G}{\partial u} + \left(\frac{\partial v}{\partial y}\right)^2 \frac{\partial^2 G}{\partial v^2} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\frac{\partial^2 G}{\partial v \partial u} + \frac{\partial^2 v}{\partial y^2}\frac{\partial G}{\partial u} \tag{Y.635}$$

These two expressions can be combined into

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \ldots = \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]\left(\frac{\partial^2 G}{\partial u^2} + \frac{\partial^2 G}{\partial v^2}\right) \tag{Y.636}$$

by making use of the interchangeability of the second partial derivatives and the Cauchy-Riemann differential equations. The prefactor in square brackets is the positive conformal factor. In a actual application the problem of performing the convolution of the Green-function with the charge distribution is then reduced to finding an analytical mapping between the simple and the complicated geometry.

Z INDEX NOTATION

Z.1 *Vectors and linear forms*

Many quantities in physics have components, or internal degrees of freedom. This is particularly true in modern physics, with e.g. the realisation that the energy density $\rho c^2 = T_t{}^t$ is part of the energy momentum-tensor $T_\mu{}^\nu$ as a larger entity. The geometric picture is that there is a (vector)-space for all vectors $\boldsymbol{\upsilon} = \upsilon^i \boldsymbol{e}_i = \sum_i \upsilon^i \boldsymbol{e}_i$ which are decomposed into their components υ^i with a basis $\boldsymbol{e}_i$, with the Einstein summation convention in place. Velocities $\boldsymbol{\upsilon}$, accelerations $\boldsymbol{a}$, the magnetic field $\mathbf{B}$ and the dielectric displacement $\mathbf{D}$ are examples of vectors. There is an associated (vector) space of linear forms $\boldsymbol{p} = p_i \boldsymbol{e}^i = \sum_i p_i \boldsymbol{e}^i$ which has identical geometric properties and is spanned by a basis $\boldsymbol{e}^i$. Examples of linear forms are, for instance, the canonical momentum $\boldsymbol{p}$, the gradient of a potential $\partial\Phi$, the electric field $\mathbf{E}$ or the magnetic induction $\mathbf{H}$.

A very useful notation used throughout theoretical physics is the so-called abstract index notation, where one works entirely with the components of vectors and linear forms, with an implicitly assumed basis. By convention, one denotes vectors with a superscript, contravariant index υ^i and linear forms with a subscript, covariant index p_j.

Canonically, one defines an orthogonality relation $\boldsymbol{e}^i \boldsymbol{e}_j = \delta^i_j$ between the basis vector of the vector space and the basis linear forms, such that the inner product between a vector $\boldsymbol{\upsilon}$ and a linear form $\boldsymbol{p}$ is given by

$$\boldsymbol{p} \cdot \boldsymbol{\upsilon} = p_i \boldsymbol{e}^i \, \upsilon^j \boldsymbol{e}_j = p_i \upsilon^j \, \boldsymbol{e}^i \boldsymbol{e}_j = p_i \upsilon^j \delta^i_j = p_i \upsilon^i. \tag{Z.637}$$

According to the Einstein sum convention (also called a contraction), an expression like $p_i \upsilon^i$ is to be interpreted as $\sum_i p_i \upsilon^i$, with an automatic implied summation over all index pairs which are appear as super- and subscripts.

A metric γ_{ij} is used for converting a vector υ^j to its associated linear form $\upsilon_i = \gamma_{ij}\upsilon^j$, while the inverse metric γ^{ij} does the opposite: It translates a linear form p_j to its associated vector $p^i = \gamma^{ij} p_j$. Of course, making a linear form out of a vector and then translating it back to being a vector again can not change anything,

$$\gamma^{ij}\left(\gamma_{jk}\upsilon^k\right) = \underbrace{\gamma^{ij}\gamma_{jk}}_{=\delta^i_k} \upsilon^k = \upsilon^i \tag{Z.638}$$

and in this sense the metric and its inverse are related to each other:

$$\gamma^{ij}\gamma_{jk} = \delta^i_k. \tag{Z.639}$$

Instead of computing $p_i \upsilon^i$ directly as the contraction between a linear form p_i and the vector υ^i, one can use the metric to generate the linear form p_i from a vector, $p_i = \gamma_{ij} p^j$ to arrive at

$$p_i \upsilon^i = \gamma_{ij} p^i \upsilon^i = \gamma^{ij} p_i \upsilon_j \tag{Z.640}$$

Alternatively, one can generate the vector $\upsilon^i = \gamma^{ij}\upsilon_j$ from the associated linear form υ_j using the inverse metric γ^{ij}. With this argument, one can say that the metric defines a scalar product between vectors, while the inverse metric defines a scalar product between linear forms. It is well worth it to differentiate carefully between the metric γ_{ij} and the Kronecker symbol δ^i_j, even in the case of Euclidean vector spaces. The Kronecker symbol renames indices of vectors or linear forms, but never changes them:

$$\upsilon^i = \delta^i_j \upsilon^j \quad \text{and} \quad p_i = \delta^j_i p_j \tag{Z.641}$$

Z.2 *Coordinates and differentials*

Coordinates are usually written as vectorial tuples x^i (in themselves, they are not vectors!), and this choice is purely conventional. The coordinates have the property that every entry of x^i can change independently from the others, so

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j \tag{Z.642}$$

because x^i only changes in the direction of x^i at unit speed, whereas x^i remains constant if x^j is changed. This is encapsulated by the Kronecker symbol δ^i_j. But in this sense, derivatives with respect to the coordinates $\partial_i = \partial/\partial x^i$ are linear forms,

$$\frac{\partial x^i}{\partial x^j} = \frac{\partial}{\partial x^j} x^i = \partial_j x^i = \delta^i_j \tag{Z.643}$$

and the contraction

$$\partial_i x^i = \frac{\partial x^i}{\partial x^i} = \delta^i_i = n \tag{Z.644}$$

is sensibly defined and returns the dimensionality n. Then, the divergence $\partial_i \upsilon^i$ of a vector υ^i is defined in a straightforward way, and the divergence of a linear form would be $\gamma^{ij}\partial_i p_j = \partial_i \gamma^{ij} p_j = \partial_i p^i$ with the inverse metric.

This point can be illustrated better by considering a curve $x^i(\lambda)$ which runs through a scalar field Φ: The derivative of Φ along the curve as λ evolves, is

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\partial \Phi}{\partial x^i} = \dot{x}^i \partial_i \Phi \tag{Z.645}$$

by virtue of the chain rule. We interpret this expression as the projection, or scalar product between the gradient $\partial_i \Phi$ of the potential as a linear form with the velocity $\dot{x}^i = \upsilon^i = \mathrm{d}x^i(\lambda)/\mathrm{d}\lambda$ as a vector.

Let's try out a change of coordinates with an invertible and differentiable replacement $x^i(y^a)$: The chain rule suggests that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\partial \Phi}{\partial x^i} = \left(\frac{\mathrm{d}y^a}{\mathrm{d}\lambda}\frac{\partial x^i}{\partial y^a}\right)\left(\frac{\partial y^b}{\partial x^i}\frac{\partial \Phi}{\partial y^b}\right) = \frac{\mathrm{d}y^a}{\mathrm{d}\lambda}\underbrace{\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^i}}_{=\delta^b_a}\frac{\partial \Phi}{\partial y^b} = \frac{\mathrm{d}y^a}{\mathrm{d}\lambda}\frac{\partial \Phi}{\partial y^a} = \dot{y}^a \partial_a \Phi \tag{Z.646}$$

such that the rate $\mathrm{d}\Phi/\mathrm{d}\lambda$ is unchanged, no matter which coordinates have been used to compute the velocity and the gradient. This is achieved because the Jacobian $\partial x^i/\partial y^a$ used to transform the vectorial velocity and $\partial y^b/\partial x^i$ for the transformation of the potential gradient as a linear form are inverses to each other:

$$\frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^i} = \frac{\partial y^b}{\partial y^a} = \delta^b_a \tag{Z.647}$$

by recognising that the expression originates from $\partial y^b/\partial y^a$ from an intermediate differentiation with respect to x^i as dictated by the chain rule. With the latter relation it becomes clear that even though the coordinates x^i are not (yet) a vector, the velocity $\upsilon^i = \mathrm{d}x^i/\mathrm{d}\lambda$ as the derivative is, and the gradient $\partial\Phi/\partial x^i$ is truly a linear form: Both have the correct transformation properties. Vectors such as the velocity transforms according to $\upsilon^i \rightarrow \mathrm{J}^i{}_a\upsilon^a = \partial x^i/\partial y^a\ \upsilon^a$, and linear forms inversely, $p_i \rightarrow \mathrm{J}^a{}_i p_a = \partial y^a/\partial x^i\ p_a$. Indeed, in differential geometry all quantities (scalars, vectors, linear forms, tensors of various rank and valence) are defined through their transformation behaviour.

The Kronecker symbol arises as the fundamental property of the coordinates y^a then makes sure that only equal indices are considered in multiplying $\dot{y}^a\delta^b_a\partial_b\Phi = \dot{y}^a\partial_a\Phi$. This neat cancellation would not automatically take place in scalar products between two vectors: Defining the Jacobian $\mathrm{J}^i{}_a = \partial x^i/\partial y^a$ suggests the transformation $\upsilon^i \rightarrow \mathrm{J}^i{}_a\upsilon^a$, and the scalar product $\gamma_{ij}\upsilon^i\upsilon^j$ can only be invariant if the metric transforms inversely (defining an orthogonal transform), $\gamma_{ij} \rightarrow \mathrm{J}_i{}^a\mathrm{J}_j{}^b\gamma_{ab}$ with the inverse Jacobian $\mathrm{J}_i{}^a$:

$$\mathrm{J}^i{}_a\mathrm{J}_i{}^b = \frac{\partial x^i}{\partial y^a}\frac{\partial y^b}{\partial x^i} = \delta^b_a \tag{Z.648}$$

such that scalar products are in fact invariant:

$$\mathrm{J}_i{}^a\mathrm{J}_j{}^b\gamma_{ab}\ \mathrm{J}^i{}_c\upsilon^c\ \mathrm{J}^j{}_d\upsilon^d = \mathrm{J}_i{}^a\mathrm{J}^i{}_c\ \mathrm{J}_j{}^b\mathrm{J}^j{}_d\ \gamma_{ab}\upsilon^c\upsilon^d = \delta^a_c\delta^b_d\gamma_{ab}\upsilon^c\upsilon^d = \gamma_{ab}\ \delta^a_c\upsilon^c\ \delta^b_d\upsilon^d = \gamma_{ab}\upsilon^a\upsilon^b. \tag{Z.649}$$

The same argument applies to the invariance of the scalar product $\gamma^{ij}p_ip_j$, only that the Jacobians now transforms the inverse metric γ^{ij} and the inverse Jacobians the linear forms p_i:

$$\mathrm{J}^i{}_a\mathrm{J}^j{}_b\gamma^{ab}\mathrm{J}_i{}^c p_c\mathrm{J}_j{}^d p_d = \mathrm{J}^i{}_a\mathrm{J}_i{}^c\ \mathrm{J}^j{}_b\mathrm{J}_j{}^d\ \gamma^{ab}p_cp_d = \delta^c_a\delta^d_b\gamma^{ab}p_cp_d = \gamma^{ab}\ \delta^c_a p_c\ \delta^d_b p_d = \gamma^{ab}p_ap_b \tag{Z.650}$$

The transformation properties of the metric and its inverse show that they are in fact tensors of rank 2.

Z.3 *Lagrange- and Hamilton-formalism in components*

If one chooses the coordinates to be summarised in a vectorial tuple x^i, the velocity $\dot{x}^i = \mathrm{d}x^i/\mathrm{d}t$ and the acceleration $\ddot{x}^i = \mathrm{d}^2x^i/\mathrm{d}t^2$ are vectors as well. The construction of a scalar quantity like the Lagrange function requires the metric γ_{ij} for the kinetic term,

$$\mathcal{L}(x^i, \dot{x}^i) = \frac{m}{2}\gamma_{ij}\dot{x}^i\dot{x}^j - \Phi(x^i) \tag{Z.651}$$

as well as the potential Φ. Variation with the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{x}^a} = \frac{\partial\mathcal{L}}{\partial x^a} \tag{Z.652}$$

leads to

$$\frac{\partial \mathrm{L}}{\partial\dot{x}^a} = \frac{m}{2}\gamma_{ij}\Big(\underbrace{\frac{\partial\dot{x}^i}{\partial\dot{x}^a}}_{=\delta^i_a}\dot{x}^j + \dot{x}^i\underbrace{\frac{\partial\dot{x}^j}{\partial\dot{x}^a}}_{=\delta^j_a}\Big) = \frac{m}{2}\Big(\gamma_{aj}\dot{x}^j + \gamma_{ia}\dot{x}^i\Big) = m\gamma_{aj}\dot{x}^j \tag{Z.653}$$

because the metric is symmetric, $\gamma_{ia} = \gamma_{ai}$, and any internal index in an expression can be renamed. Together with

$$\frac{\partial\mathcal{L}}{\partial x^a} = -\frac{\partial\Phi}{\partial x^a} \tag{Z.654}$$

one arrives at the Newtonian equation of motion

$$m\gamma_{aj}\ddot{x}^j = -\frac{\partial\Phi}{\partial x^a} \tag{Z.655}$$

which can be brought into a more familiar shape by multiplying both sides with the inverse metric γ^{ia}:

$$m\gamma^{ia}\gamma_{aj}\ddot{x}^j = m\delta^i_j\ddot{x}^j = m\ddot{x}^i = -\gamma^{ia}\frac{\partial\Phi}{\partial x^a} = -\gamma^{ia}\partial_a\Phi \quad\rightarrow\quad m\ddot{x}^i = -\gamma^{ia}\partial_a\Phi \tag{Z.656}$$

with $\gamma^{ia}\gamma_{aj} = \delta^i_j$ such that the inverse metric relates the gradient of the potential, itself a linear form, to the acceleration as a vector.

The canonical momentum,

$$p_i = \frac{\partial\mathcal{L}}{\partial\dot{x}^i} \tag{Z.657}$$

is, by this reasoning, a linear form (and a function $p_j(\dot{x}^i)$ of the vectorial velocity $\dot{x}^i$, which can be inverted to yield $\dot{x}^i(p_j)$ for convex Lagrange-functions), so that the Legendre transform

$$\mathcal{H} = p_i\dot{x}^i(p_j) - \mathcal{L}(x^i, \dot{x}^i(p_j)) \tag{Z.658}$$

is sensibly defined and yields a scalar Hamilton function. The contraction of the vectorial velocity $\dot{x}^i$ with the linear form p_i appears naturally. And it provides an argument, why the canonical momentum $p_i = \partial\mathcal{L}/\partial\dot{x}^i$ is more than just the kinetic momentum $m\dot{x}^i$: On the contrary, with the definition of the canonical momentum p_i one obtains for a standard form of the Lagrange-function

$$p_i = m\gamma_{ij}\dot{x}^i \quad \text{and consequently,} \quad \dot{p}_i = -\partial_i\Phi \tag{Z.659}$$

from the Euler-Lagrange equation, showing how the metric is necessary, in one way or another, to mediate between velocity and acceleration as vectorial quantities on one side and momentum and potential gradient as linear forms on the other, even in the case of a standard kinetic term in the Lagrange-function. Hamilton's equations of motion

$$\dot{p}_i = -\frac{\partial\mathcal{H}}{\partial x^i} \quad \text{and} \quad \dot{x}^i = +\frac{\partial\mathcal{H}}{\partial p_i} \tag{Z.660}$$

remain consistent as the derivative with respect to a vector is a linear form, while the derivative with respect to a linear form returns again a vector: $\partial p_i/\partial p_j = \delta_i^j$ for p_i as a phase space coordinate. Please note how $\dot{p}_i$ as a linear form emerges from $-\partial\mathrm{H}/\partial x^i = -\partial\Phi/\partial x^i$ without a metric in contrast to equation (Z.655), in a consistent variant of Newton's second law: $\dot{p}_i = -\partial_i\Phi$.

Z.4 *Duals*

The cross product $\boldsymbol{x}\times\boldsymbol{y}$ between two vectors is defined in terms of their basis decomposition as

$$\boldsymbol{x}\times\boldsymbol{y} = x^j\boldsymbol{e}_j\times y^k\boldsymbol{e}_k = x^jy^k\,\boldsymbol{e}_j\times\boldsymbol{e}_k = x^jy^k\,\epsilon_{ijk}\boldsymbol{e}^i = \underbrace{\epsilon_{ijk}x^jy^k}_{=(\boldsymbol{x}\times\boldsymbol{y})_i}\boldsymbol{e}^i, \tag{Z.661}$$

with the Levi-Civita symbol as an expression of the right-handed orientation of the (orthogonal) basis system. Therefore, cross product $\epsilon_{ijk}x^jy^k$ is naturally a linear form, but is it possible to construct a naturally antisymmetric quantity out of the vectors x^j and y^k as a vectorial object? Clearly, the antisymmetric rank-2 tensor $x^jy^k - x^ky^j$ would be such a thing, and would be, up to a factor of two, equal to the cross product:

$$\epsilon_{ijk}\left(x^jy^k - x^ky^j\right) = \epsilon_{ijk}x^jy^k - \epsilon_{ikj}x^jy^k = \left(\epsilon_{ijk} - \epsilon_{ikj}\right)x^jy^k = 2\epsilon_{ijk}x^jy^k \tag{Z.662}$$

where in the first step the indices are interchanged $j \leftrightarrow k$, and then the property $\epsilon_{ijk} = -\epsilon_{ikj}$ is used. $(x^jy^k - x^ky^j)/2$ is called the dual, and the usability hinges heavily on the fact that the contraction of two antisymmetric objects is nonzero. The dual $x^jy^k - x^ky^j$ is a vectorial (antisymmetric) tensor that contains the same information as the linear form resulting from $\epsilon_{ijk}x^jy^k$. Duals can be defined for any antisymmetric tensor, for instance $\tilde{\mathrm{G}}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}\mathrm{G}^{\mu\nu}$. They convert all Maxwell-equations into divergences, as

$$\epsilon^{ijk}\partial_j\mathrm{E}_k = \partial_j\left(\epsilon^{ijk}\mathrm{E}_k\right) = \partial_j\mathrm{E}^{ij} = -\partial_{ct}\mathrm{B}^i, \tag{Z.663}$$

as exemplified by the induction equation.

Z.5 *Gauß- and Stokes-theorems*

The Gauß-theorem relates the volume integral over the divergence of a vector field to the integral of that particular vector field over the surface bounding the volume,

$$\int_V \mathrm{dV}\,\partial_i \mathrm{D}^i = \int_{\partial V} \mathrm{dS}_i\ \mathrm{D}^i \quad \text{and} \quad \int_V \mathrm{dV}\,\partial_i \mathrm{B}^i = \int_{\partial V} \mathrm{dS}_i\ \mathrm{B}^i, \tag{Z.664}$$

where in electrodynamics the relation gets applied to the two vector fields D^i and B^i. The surface element dS_i is a linear form, because it originates from the cross product of two vectors. Similarly, the Stokes-theorem relates the surface integral of the rotation of a field to the line integral along the boundary,

$$\int_S \mathrm{dS}_i\ \epsilon^{ijk}\partial_j \mathrm{E}_k = \int_{\partial S} \mathrm{d}r^i\ \mathrm{E}_i \quad \text{and} \quad \int_S \mathrm{dS}_i\ \epsilon^{ijk}\partial_j \mathrm{H}_k = \int_{\partial S} \mathrm{d}r^i\ \mathrm{H}_i, \tag{Z.665}$$

where in electrodynamics this becomes relevant for the two linear forms E_i and H_i. It is a bit remarkable that the assignment of vectors and linear forms to the fields in Maxwell's equations only needs as geometric objects the differential ∂_i and the associated surface element dS_i as linear forms, and never their possible vectorial counterparts. The Gauß-theorem gets only ever applied to the vectors D^i and B^i, whereas the application of the Stokes-theorem is restricted to the linear forms E_i and H_i. This, in fact, is a hint that electrodynamics would work even on non-metric spacetimes, because the metric (and its inverse) would be a mean to convert between the two types of fields.

Z.6 *Summary of co- and contravariant quantities in electrodynamics*

0. rank 0: scalars and pseudoscalars

Φ	electric potential
θ	axion field amplitude
ρ	electric charge density
dV	volume element

1. rank 1: vectors and linear forms

x^i	Euclidean coordinates	∂_i	coordinate differential
$\dot{x}^i$	velocity	p_i	momentum
$\ddot{x}^i$	acceleration	$\partial_i\Phi$	potential gradient
D^i	dielectric displacement	E_i	electric field
B^i	magnetic field	H_i	magnetic induction
		A_i	vector potential
P^i	Poynting vector	Y_i	Poynting linear form
$\jmath^i$	electric current density	dS_i	surface element

x^μ	Minkowski coordinates	∂_μ	coordinate differential
u^μ	4-velocity	p_μ	4-momentum
		A_μ	4-potential
$\jmath^\mu$	4-current density		

2. rank 2: co-, contravariant and mixed tensors

γ^{ij}	inverse Euclidean metric	γ_{ij}	Euclidean metric
ϵ^{ij}	permissivity tensor	ϵ_{ij}	inverse permissivity
μ^{ij}	permeability tensor	μ_{ij}	inverse permeability
σ^{ij}	conductivity		

$\eta^{\mu\nu}$	inverse Minkowski metric	$\eta_{\mu\nu}$	Minkowski metric
$G^{\mu\nu}$	excitation	$F_{\mu\nu}$	Faraday tensor
$\tilde{F}^{\mu\nu}$	Faraday dual	$\tilde{G}_{\mu\nu}$	excitation dual

δ_i^j, δ_ν^μ	Kronecker-symbol
$T_i^{\ j}$	Maxwell stress tensor
$T_\mu^{\ \nu}$	energy-momentum tensor
$\Lambda^i_{\ j}$	endomorphism for vectors $v^i \to \Lambda^i_{\ j} v^j$
$\Lambda_i^{\ j}$	endomorphisms for linear forms $p_i \to \Lambda_i^{\ j} p_j$

ACKNOWLEDGEMENTS

I am indebted to Nils Schmitt, who helped with the typesetting of the script, and Rebecca Maria Kuntz as well as Heinrich von Campe for their support in writing, in particular with notational problems, and with their aesthetic attitude. Martin Bauer, Anja Butter, Rebecca Maria Kuntz and Heinrich von Campe supported the lecture as tutors, for which I am very grateful. I would like to express my gratitude to Jörg Jäckel, for discussions in particular in relation to axions, and to Francesco Pace. And of course I would like to thank my students Zichu Li (violin), Vivienne Leidel (clarinet), Britta Zieser (human voice, soprano), Sebastian Leyer (trumpet, tuba, trombone), Luca Prechtel (flute, English horn, oboe, saxophone, bassoon, piccolo flute), Max Anton Gramberg (guitar), Jan Brauch (organ), Kenneth von Bünau (cello), Jan Schneider (banjo, mandolin), Mia Sittinger (recorder), Frederik Carstens (harp), Niklas Gronwald (French horn), Noah Kriesch (viola) for providing samples of their instruments for computing the Fourier-transform. The sampling and Fourier-transform was done with PhyPhox. I'm indebted to Justus Zimmermann, whose careful reading made him spot many mistakes.

There is a large number of excellent textbooks on electrodynamics and relativistic field theory, and my script is not supposed to be a replacement for them. In no particular order I would like to mention:

- J.D. Jackson: Classical Electrodynamics, Wiley, 1998
- W. Greiner: Classical Electrodynamics, Springer, 1992
- F. Scheck: Classical Field Theory, Springer, 2012
- J.D. Bjorken, S.D. Drell: Relativistic Quantum Fields, McGraw-Hill, 1965

Concerning notation in this script, the index notation and the distinction between vectors and linear forms, my readers deserve an apology, or at least a justification: My feeling was that students take some time to transition to the index notation which is widely used in field theory and relativity, and one might as well start that transition early in the curriculum. I hope that nowhere there was an unexplained or underived vector identity, which I found dissatisfying as a student and which I hope to remedy in my script. To my view, the subtleties related to vectors and linear forms, and the properties of media matter a lot, for instance the differences between energy and momentum transport, and it was my intention to convey this in my lecture.

I would like to acknowledge the lecture LaTeX-class by V.H. Belvadi. The spacetime-diagrams are adapted from the examples on TikZ.net and I am very grateful to I. Neutelings, who made this possible. All python scripts for the generation of figures are available on request, with no implied support.

INDEX

Electrodynamics is a cornerstone of every modern education in theoretical physics, as it introduces a geometric picture of the laws of Nature, and is permeated by relativity. Starting from the fundamental phenomenology of Maxwell's equations, the script treats the construction of Green-functions for solving potential problems, before moving to the dynamics of the electromagnetic field and the Poynting-theorems. Retardation bridges to the notion of light cones and the emergence of relativity, leading to a covariant formulation of Maxwell's equations. Gauge transformations are treated in detail, as well as the behaviour of Maxwell's equation under discrete symmetries.

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ISBN 978-3-96822-170-0
9 783968 221700

27,00 EUR (DE)
27,80 EUR (AT)